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Paper:

Consistent Conjectural Variations Equilibrium in a Semi-Mixed Duopoly

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This paper considers conjectural variations equilibrium (CVE) in the one item market with a mixed duopoly of competitors. The duopoly is called *semi-mixed* because one (semi-public) company's objective is to maximize a convex combination of her net profit and domestic social surplus (DSS). The two agents make conjectures about fluctuations of the equilibrium price occurring after their supplies having been varied. Based on the concepts of the *exterior* and *interior equilibrium*, as well as the existence theorem for the interior equilibrium (a.k.a. the consistent CVE, or the exterior equilibrium with *consistent conjectures*) demonstrated in the authors' previous papers, we analyze the behavior of the interior equilibrium as a function of the semi-public firm's level of socialization. When this parameter reflected by the convex combination coefficient tends to 1, thus transforming the semi-public company into a completely public one, and the considered model into the classical mixed duopoly, two trends are apparent. First, for the private company, the equilibrium with consistent conjectures (CCVE) becomes more attractive (lucrative) than the Cournot-Nash equilibrium. Second, there exists a (unique in the case of an affine demand function) value of the convex combination coefficient such that the private agent's profit is the same in both of the above-mentioned equilibrium types, thus making no subsidy to the producer or to the consumers necessary. Numerical experiments with various mixed duopoly models confirm the robustness of the proposed algorithm for finding the optimal value of the above-mentioned combination coefficient (a.k.a. the semi-public company's socialization level).

Keywords: game theory, mixed duopoly, conjectural variations equilibrium (CVE), consistent conjectures, optimal socialization level

1. Introduction

During the last 15 years, researchers in the field of mathematical economics have extensively and intensively studied mixed oligopolies models. In contrast to the classical oligopoly, a mixed oligopoly, apart from standard producers who seek to maximize their net profit, usually includes (at least one) public company trying to optimize another objective function involving indicators of the firm's social responsibility. Many such models include an agent who maximizes the domestic social surplus (cf., [1–5]). An income-per-worker function replaces the standard net profit objective function in some other publications (cf., [6–9]). Other researchers [10, 11] have studied a third kind of mixed duopoly, in which an exclusive participant aims to maximize a convex combination of his/her net profit and domestic social surplus. This paper addresses such a company as *semi-public*.

In many of the aforementioned works, the authors investigated mixed oligopolies by making use of the classical Cournot-Nash, Hotelling, or Stackelberg models. The notion of *conjectural variations equilibrium* (CVE) first introduced by Bowley [12] and Frisch [13] opens another way of the agents' reaction to the market challenge, which attracts ever-growing interest on part of the related researchers. In CVE, competitors behave as follows – each agent (producer) selects her/his most favorable strategy having supposed that every opponent's action is a *conjec-*

tural variation function of her/his own strategical variation. For example, as Laitner (p. 643 of [14]) states, "Although the firms make their output decisions simultaneously, plan changes are always possible before production begins." In other words, in contrast to the Cournot-Nash approach, here, every company assumes that its choice of the own output volume will affect her competitors' reaction. The consequently arising prediction (or, conjectural variation) function is the central point of conjectural variation decision making, or the conjectural variations equilibrium (CVE).

As is mentioned in [15] and [16], the notion of CVE has been the topic of abundant theoretical discussions (cf., [17]). Notwithstanding this, economists have extensively used various forms of CVE to predict the outcome of non-cooperative behavior in many areas of economics. The literature on conjectural variations has focused mainly on two-player games (cf., [15]) because a serious conceptual difficulty arises if the number of agents is greater than two (cf., [15] and [18]).

In order to overcome conceptual hurdles arising in many-player games, a new tool was developed in [19]; namely, instead of imposing very restrictive additional assumptions (like the identity of players in the oligopoly), it was supposed that every player makes conjectures only about the variations of the market clearing (equilibrium) price as a consequence of (infinitesimal) variations of the same player's output volume. Knowing the opponents' conjectures (the *influence coefficients*), each firm applies a verification procedure in order to determine whether its influence coefficient is *consistent* with those of the remaining agents.

In papers [18] and [20], the authors extended the ideas of [19] to the mixed duopoly and oligopoly cases, respectively. They defined *exterior equilibrium* as a CVE state with the conjectures fixed in an extrinsic manner. This sort of CVE was proved to exist uniquely, which helped introduce the notion of *interior equilibrium* as the exterior equilibrium with consistent conjectures (influence coefficients). All these instruments – the consistency criteria, consistency verification procedures, and existence theorems for the interior equilibrium were developed and demonstrated in [18] and [20].

In the next series of papers, namely, [21, 22], the aforesaid constructions were extended to the case of a semi-mixed duopoly, where similar to [10] and [11], the (semi-) public company strives to maximize a convex combination of the net profit and domestic social surplus. The results of numerical experiments with a test model of a market of electricity resembling that of [23], both with and without a semi-public producer in the set of agents, showed that the consumer gains more if the semi-public agent follows the CVE strategy as compared to the Nash-Cournot equilibrium. Furthermore, in [22], the authors declared a guess that there must exist such a value of the combination parameter (also interpreted as the public firm's socialization level) that brings up the "equivalence" (in a certain sense) of the consistent conjectural variations equilibrium (CCVE) and the classical Cournot-Nash one. This equivalence permits a socially responsible municipality to diminish (cancel) subsidies paid either to the private company (in order to compensate its losses when following the consistent conjectures), or to the consumers (to reimburse them the higher retail price of the good if both the competing semi-public firm and the private company both are stuck to the Cournot-Nash conjectures).

In this paper, we present mathematically rigorous proofs of the above-mentioned guess. In other words, we establish the existence of the value of the combination coefficient (also known as the semi-public enterprise's socialization level) such that the private agent's profit is the same in the CCVE and Cournot-Nash equilibrium states, which makes the subsidies from the authorities either to the producer or to the consumers unnecessary.

The rest of the paper is organized as follows. Section 2 formulates the mixed duopoly model and the two kinds of equilibrium we consider (exterior and interior). In addition, we present the main theorems showing the existence and uniqueness of the exterior equilibrium for any set of feasible conjectures (influence coefficients), as well as the formulas for the derivative of the equilibrium price, p, with respect to the active demand variable, D. Moreover, Subsection 2.2 deals with the consistency criterion and the definition of the interior equilibrium (which can be treated as a consistent CVE state, or CCVE); the CCVE existence theorems from [21] and [22] are also discussed. In Section 3, we consider an important particular case when the demand function is linear (more exactly, affine). This stronger assumption allows one not only to prove the existence and uniqueness of the interior (consistent) CVE, but also to investigate the changing of the optimal function values of both producers as the function of the convex combination coefficient (the semi-public firm's socialization level) in Subsections 3.1 through 3.3. Some elements of the comparative statics concerning the three most popular equilibrium sorts (CCVE, Cournot-Nash, and the perfect competition) are also provided in Subsection 3.4. Subsection 3.5 introduces the optimality criterion for the convex combination coefficient (the semi-public firm's socialization level) and establishes the existence of its optimal value in the interval $[0,1]$. Section 4 presents the results of the numerical experiments with several examples of the semi-mixed duopoly, illustrating the importance of finding the optimal socialization parameter value. Conclusions and future research plans are discussed in Section 5, while the Acknowledgments and Reference list finish the main body of the paper. All the proofs of principal theorems (being long and complicated) are exported to the Appendices.

2. Model Specification

We will examine a semi-mixed duopoly with two agents numbered as follows: $i = 0$ is a semi-public company and $i = 1$ is a private firm. The companies supply a homogeneous produce under the expenditure estimated by the *cost functions* $f_i(q_i)$, $i = 0,1$, where $q_i \geq 0$ is the *output volume* by agent i. The market clearing supply is specified by a demand (inverse price) function $G = G(p)$, the argument p of which is the price suggested by the suppliers. A (fixed) amount of the active demand, D, is nonnegative and is independent of the price. The equilibrium between the demand and supply for a given price p is reflected in the following (balance) equality:

$$q_0 + q_1 = G(p) + D. \qquad (1)$$

Furthermore, the private supplier, $i = 1$, selects his output, $q_1 \geq 0$, in order to maximize his (net) profit function,

$$\pi_1(p, q_1) = p \cdot q_1 - f_1(q_1), \qquad (2)$$

whereas the semi-public company, $i = 0$, decides the volume of its output, $q_0 \geq 0$, with the aim of maximizing the convex combination of *domestic social surplus* and the (net) profit function,

$$S(p, q_0, q_1) = \beta \left(\int_0^{q_0+q_1} p(x)dx - p \cdot q_1 - f_0(q_0) \right) + (1-\beta)(p \cdot q_0 - f_0(q_0)), \qquad (3)$$

where $0 < \beta \leq 1$. Here, domestic social surplus involving the integral in Eq. (3) is usually interpreted as the money gained by the (domestic) consumer when he/she acquires the good at the lower price (established in the market) than that expected by him/her *before* the semi-public company entered the market (*see* the more detailed interpretation by the well-known Japanese mathematicians and economists [3–5]).

In order to describe our model in rigorous mathematical terms, we suppose that the model boasts the following properties.

A1. *The demand (inverse price) function $G = G(p) \geq 0$ has finite values for all $p \geq 0$, and is continuously differentiable with $G'(p) \leq 0$.*

A2. *For each $i = 0, 1$, the cost function $f_i(q_i)$ is quadratic with zero overhead costs, i.e.,*

$$f_i(q_i) = \frac{1}{2} a_i q_i^2 + b_i q_i, \qquad (4)$$

where

$$a_i > 0,\ b_i > 0,\ i = 0, 1.$$

Moreover, it is assumed that

$$b_0 \leq b_1. \qquad (5)$$

According to our concept of conjectural variations equilibrium (CVE), we assume that both agents (semi-public and private) *conjecture* about variations in the clearing market price, p, as a function of the perturbations in their output quantities. In the terms of the first derivatives, the latter assumption might be described by a *conjectured* dependency of (infinitesimal) affine variations of the price, p, upon (infinitely small) perturbations of the supply quantities, q_i. Within this framework, the first order optimum condition depicting equilibrium reduces to the form: for the semi-public company ($i = 0$)

$$\frac{\partial S}{\partial q_0} = p - [\beta q_1 - (1-\beta) q_0] \frac{\partial p}{\partial q_0} - f_0'(q_0) \begin{cases} = 0, & \text{if } q_0 > 0; \\ \leq 0, & \text{if } q_0 = 0. \end{cases} \qquad (6)$$

A similar first order optimality condition for the private agent ($i = 1$) yields:

$$\frac{\partial \pi_1}{\partial q_1} = p + q_1 \frac{\partial p}{\partial q_1} - f_1'(q_1) \begin{cases} = 0, & \text{if } q_1 > 0; \\ \leq 0, & \text{if } q_1 = 0. \end{cases} \qquad (7)$$

On that account, in order to predict the (instantaneous) behavior of supplier i, one need make use of the first order derivative $\partial p / \partial q_i \equiv -v_i$ rather than the (exact) functional dependency of p on q_i. Even more, the latter dependency is extremely hard to estimate in a many-person game with several decision makers. Here, the negative sign is applied in order to have nonnegative values of v_i. Surely, the conjectured (first-order) dependency of p on q_i should guarantee the concavity of the i-th producer's conjectured profit as a function of her supply, which implies the maximum of the agent's revenue. Under the assumption that the cost functions, f_i, are quadratic and strictly convex, the concavity of the product, $p \cdot q_i$, by q_i would be enough. For example, it suffices supposing the coefficient, v_i (referred to as the i-th company's *influence coefficient*), to be nonnegative and constant. In this case, the conjectured (instantaneous first order) dependency of the private company's profit variations upon the production output η_1 has the form $[p - v_1(\eta_1 - q_1)]\eta_1 - f_1(\eta_1)$, while the (local) maximum condition at $\eta_1 = q_1$ is expressed by the relation:

$$\begin{cases} p = v_1 q_1 + a_1 q_1 + b_1, & \text{if } q_1 > 0; \\ p \leq b_1, & \text{if } q_1 = 0. \end{cases} \qquad (8)$$

Likewise, the semi-public company presumes a local dependency of domestic social surplus on her supply level η_0 in the form

$$\beta \left(\int_0^{\eta_0+q_1} p(x)dx - [p - v_0(\eta_0 - q_0)] q_1 - f_0(\eta_0) \right) + (1-\beta)\{[p - v_0(\eta_0 - q_0)]\eta_0 - f_0(\eta_0)\}. \qquad (9)$$

The latter permits formulating the maximum condition at $\eta_0 = q_0$ as follows:

$$\begin{cases} p = -v_0 [\beta q_1 - (1-\beta) q_0] + a_0 q_0 + b_0, & q_0 > 0; \\ p \leq -\beta v_0 q_1 + b_0, & q_0 = 0. \end{cases} \qquad (10)$$

If the suppliers' conjectures concerning market clearing price were assigned externally – like it was done in the previous models studied by the authors – one might assume the values, v_i, to be functions of q_i and p. Notwithstanding that, here, we exploit the framework of paper [19], wherein the conjectures in the equilibrium are calculated *concurrently* with the clearing price, p, and the supply quantities, q_i, by a particular validation routine. In this circumstance, the influence coefficients are the solutions of a nonlinear system of equations found for the equilibrium only. In Subsection 2.2, such an equilibrium state is called referred to as *interior* being in fact the extended vector comprising both the variables and parameters $(p;q_0,q_1;\nu_0,\nu_1)$. However, in order to describe the validation technique, one needs first define a more elementary concept of equilibrium referred to as *exterior* (cf., [19]) with the parameters, ν_i, assigned externally. That notion is introduced and studied in Subsection 2.1.

2.1. Exterior Equilibrium

The concept of *exterior equilibrium* in our framework can be introduced as follows.

Definition 1. A vector $(p;q_0,q_1)$ is named *exterior equilibrium* for the fixed coefficients, $\nu_i \geq 0$, $i=0,1$, whenever the market is leveled; that is, equality (1) is true, and the optimality conditions for both the private and semi-public companies (8) and (10), respectively, hold.

In what follows, we will examine solely the instance wherein the array of suppliers with positive output volumes is irrevocable, that is, it does *not* react to the values ν_i, $i=0,1$, of the influence coefficients. In order to assure this feature, an extra postulate is accepted.

A3. *For $p_0=b_1$, the ensuing inequality applies:*

$$\frac{p_0-b_0}{a_0}<G(p_0). \qquad (11)$$

Lemma 1. *Assumptions **A1**, **A2**, and **A3** entail that, for all nonnegative values of ν_i, $i=0,1$, supply values, q_i, are strictly positive at any exterior equilibrium (i.e., $q_i>0$, $i=0,1$) if, and only if $p>p_0$.*

Proof. Indeed, if $p>p_0=b_1$ then the inequalities $p\leq b_1$ and $p\leq -\beta\nu_0 q_1+b_0$, from the optimality conditions (8) and (10), respectively, never apply, which implies that no (equilibrium) value q_i, $i=0,1$, can vanish.

Conversely, if all the equilibrium outputs are positive, i.e., $q_i>0$, $i=0,1$, then the optimality condition (8) directly entails $p=\nu_1 q_1+a_1 q_1+b_1>b_1$. Hence, $p>p_0=b_1$, and the proof is complete. The next assertions are established in the Appendix.

Theorem 1. *With postulates **A1**, **A2**, and **A3**, for any $D\geq 0$, $\nu_1\geq 0$, $i=0,1$, there exists uniquely the exterior equilibrium $(p^*;q_0^*,q_1^*)$ depending continuously on the parameters $(D;\nu_0,\nu_1)$. The equilibrium price $p^*=p^*(D;\nu_0,\nu_1)$, being a function of those constants, is differentiable by D and ν_i, $i=0,1$. Furthermore, $p^*(D;\nu_0,\nu_1)>p_0$, and*

$$\frac{\partial p^*}{\partial D}=\frac{1}{\dfrac{1}{(1-\beta)\nu_0+a_0}+\dfrac{\nu_0+a_0}{(1-\beta)\nu_0+a_0}\left(\dfrac{1}{\nu_1+a_1}\right)-G'(p^*)}. \qquad (12)$$

Proof. *See* Appendix A.1.

2.2. Interior Equilibrium

Now, one can introduce the notion of interior equilibrium. First, we outline the routine of validation of the influence coefficients, ν_i, just as it was proposed in [19]. Presume that our agents reach the exterior equilibrium $(p;q_0,q_1)$ determined (uniquely) for some assigned ν_i, $i=0,1$, and D. Suppose that one of the suppliers – for example, k, $0\leq k\leq 1$ – provisionally mutate her demeanor as follows: he/she stops maximizing the conjectured profit (or domestic social surplus, as it is in case $k=0$) and makes tiny oscillations near her output, q_k. In formulas, it equals considering the list of agents reduced to the array $I_{-k}:=\{i\,:\,0\leq i\leq 1,\ i\neq k\}$ with the supply, q_k, extracted from the active demand.

Wavering the supply by company k is then tantamount to the active demand vacillation in the mode $\delta D_k:=\delta(D-q_k)=-\delta q_k$. If one assumes these oscillations to be negligible, it is logical to accept that by perceiving the respective variations in the monopoly price dictated by the other agent, company k might estimate the derivative of the (equilibrium) price with respect to the active demand, which coincides with its influence coefficient.

Employing Eq. (12) from **Theorem 1** to compute the derivative, it is necessary to take into account that supplier k is, for the present, excluded from the equilibrium model. Therefore, the term concerning $i=k$ must be kept out the denominator. Remembering that, one brings about the next condition.

Consistency Criterion. *In the exterior equilibrium $(p;q_0,q_1)$, the influence coefficients, ν_i, $i=0,1$, are referred to as **consistent** whenever the equalities below are valid:*

$$\nu_0=\frac{1}{\dfrac{1}{\nu_1+a_1}-G'(p)}, \qquad (13)$$

and

$$\nu_1=\frac{1}{\dfrac{1}{(1-\beta)\nu_0+a_0}-G'(p)}. \qquad (14)$$

Remark 1. When the two suppliers play as profit-increasing (private) firms, then Eqs. (13) and (14) shrink to a (single) rule:

$$\nu_i=\frac{1}{\sum\limits_{j\neq i}\dfrac{1}{\nu_j+a_j}-G'(p)},\ i=0,1.$$

Theorem 2. *If assumptions **A1**, **A2**, and **A3** hold, there exist interior equilibria.*

Proof. *See* Appendix A.3.

Next, express the demand function's derivative with $\tau=G'(p)$, and replace the consistency conditions (13) and (14) with the formulas below:

$$\nu_0=\frac{1}{\dfrac{1}{\nu_1+a_1}-\tau} \qquad (15)$$

and

$$v_1 = \frac{1}{\dfrac{1}{(1-\beta)v_0 + a_0} - \tau}, \quad \ldots \ldots \ldots (16)$$

where $\tau \in (-\infty, 0]$. If $\tau \to -\infty$ the solutions of system (15) and (16) tend to the (unique) limit, $v_i = 0$, $i = 0,1$. For any bounded τ, one may prove the subsequent assertion.

Theorem 3. *For all τ, there exists a unique solution, $v_i = v_i(\tau)$, $i = 0,1$, of system* (15) *and* (16)*, which continuously depends on τ. In addition, $v_i(\tau) \to 0$ whenever $\tau \to -\infty$, and $v_i(\tau)$ strictly grows and tends to $v_i(0)$ as $\tau \to 0$, $i = 0,1$.*

Proof. *See* Appendix A.2.

3. A Special Case: A Linear (Affine) Demand Function

Assume that the demand function $G(p)$ is linear (affine); that is,

$$G(p) := -Kp + T, \quad \ldots \ldots \ldots \ldots (17)$$

where, $K > 0$, $T > 0$.

Under this extra assumption, Theorem 2 entails the undermentioned corollary.

Corollary 1. *When conditions **A1**, **A2**, and **A3** are valid, then the demand function of type* (17) *for all $\beta \in (0,1]$ implies the existence of the (unique) interior equilibrium.*

Proof. *See* Appendix A.4.

This section mainly targets at the study of the behavior (as a function of the parameter β) of the three most popular equilibrium kinds: 1) the consistent conjectural variations equilibrium (CCVE), 2) the Cournot-Nash equilibrium, and 3) the perfect competition equilibrium.

3.1. Consistent Conjectural Variations Equilibrium

Recollect that the conjectural variation equilibrium (CVE) is called *consistent* if the influence coefficients at the interior CVE meet the consistency principle represented by systems (13) and (14).

It is worthy to mention that **Corollary 1**, for all $\beta \in (0,1]$, provides for the existence of the (unique) interior equilibrium $(p^*(\beta), q_0^*(\beta), q_1^*(\beta), v_0^*(\beta), v_1^*(\beta))$. Moreover, the next result is valid:

Theorem 4. *For the linear (affine) demand function $G(p)$ from Eq.* (17)*, the price $p^*(\beta)$, the outputs $q_i^*(\beta)$, $i = 0,1$, and the influence coefficients $v_i^*(\beta)$, $i = 0,1$, characterizing the (unique) interior equilibrium, together with the total market supply $G^*(\beta) = q_0^*(\beta) + q_1^*(\beta)$, are continuously differentiable by $\beta \in (0,1]$. Furthermore, $q_0^*(\beta)$ and $G^*(\beta)$ are strictly growing, whereas $p^*(\beta), v_0^*(\beta), v_1^*(\beta)$, and $q_1^*(\beta)$ strictly decrease.*

Proof. *See* Appendix B.1.

3.2. Cournot-Nash Equilibrium

Below, we will examine the comportment of the (exterior) Cournot-Nash equilibrium as a function of the parameter β.

The well-known Cournot-Nash conjecture, – that is, $\omega_i = \partial G/\partial q_i = 1$, $i = 0,1$, – in the proposed framework is equivalent to the next conjecture: $v_i = -\partial p/\partial q_i = -1/G'(p) = 1/K$, $i = 0,1$. For all $\beta \in (0,1]$, **Theorem 1** implies that there exists (uniquely) the exterior Cournot-Nash equilibrium denoted by $(p^c(\beta), q_0^c(\beta), q_1^c(\beta))$. For matching the Cournot-Nash equilibrium to the consistent CVE, the undermentioned theorem is indispensable.

Theorem 5. *For the linear (affine) demand function $G(p)$ described in Eq.* (17)*, the price $p^c(\beta)$ and the supply values $q_i^c(\beta)$, $i = 0,1$, from the Cournot-Nash equilibrium, are continuously differentiable with respect to $\beta \in (0,1]$. Furthermore, $p^c(\beta)$ and $q_1^c(\beta)$ strictly decrease, whereas $q_0^c(\beta)$ strictly grows along with β.*

Proof. *See* Appendix B.2.

Remark 2. It is quite evident that the Cournot-Nash equilibrium in our framework need not mandatory meet the consistency rule; that is, it is most often *not* interior (i.e., *inconsistent*) equilibrium.

3.3. Perfect Competition Equilibrium

Eventually, the comportment of the (exterior) perfect competition equilibrium as a function of the parameter β will be evaluated.

The perfect competition conjecture – that is, $\omega_i = 0$, $i = 0,1$, – in our framework is described with the subsequent conjecture: $v_i = -\partial p/\partial q_i = 0$, $i = 0,1$. For every $\beta \in (0,1]$, **Theorem 1** guarantees that there exists uniquely the exterior equilibrium implementing the perfect competition. The latter will be represented by $(p^t(\beta), q_0^t(\beta), q_1^t(\beta))$. Once more, in order to conduct a comparative study of all three types of equilibrium in our linear (affine) framework we will establish the undermentioned clear-cut formulas for the perfect equilibrium supplies and price.

Theorem 6. *If the demand function $G(p)$ is depicted as in Eq.* (17)*, the price $p^t(\beta)$ and the output volumes $q_i^t(\beta)$, $i = 0,1$, related to the perfect competition equilibrium, are invariant for all $\beta \in (0,1]$ and are described by the clear-cut expressions:*

$$p^t = \frac{a_0 b_1 + a_1 b_0 + a_0 a_1 (T + D)}{a_0 + a_1 + a_0 a_1 K}; \quad \ldots \ldots (18)$$

$$q_0^t = \frac{a_1 (G(b_0) + D) + (b_1 - b_0)}{a_0 + a_1 + a_0 a_1 K}; \quad \ldots \ldots (19)$$

$$q_1^t = \frac{a_0 (G(b_1) + D) - (b_1 - b_0)}{a_0 + a_1 + a_0 a_1 K}. \quad \ldots \ldots (20)$$

Proof. *See* Appendix B.3.

Remark 3. Like in the Cournot-Nash case, one can easily see that the perfect competition equilibrium within our framework does *not* mandatory meet the consistency conditions; that is, it is *non*-interior (i.e., *inconsistent*) equilibrium.

3.4. Comparison of Consistent CVE with Cournot-Nash and Perfect Competition Equilibria

In this subsection, taking an advantage of the simple (affine) demand function, we will deduce some comparative statics results. These are always interesting for evaluating the strong and weak points of different concepts of more or less similar nature.

Theorem 7. *For the linear (affine) function, $G(p)$, described in Eq.* (17)*, the price functions in the consistent CVE, $p^*(\beta)$, the Cournot-Nash equilibrium, $p^c(\beta)$, and the perfect competition equilibrium, p^t, satisfy the following inequalities:*

$$p^t < \lim_{\beta \to 0} p^*(\beta), \quad \ldots\ldots\ldots\ldots \quad (21)$$

and

$$p^*(\beta) < p^c(\beta) \text{ for all } \beta \in (0,1]. \quad \ldots\ldots \quad (22)$$

Proof. *See* Appendix C.1.

Remark 4. First, since $v_0 = v_1 = 0$, the perfect competition equilibrium price, p^t, is the same (constant) for all $\beta \in [0,1]$. Second, inequality (21), in general, does not hold when $\beta \to 1$. The latter is a very curious result because the perfect competition equilibrium price p^t is usually the lowest in the market. Moreover, in some cases, it may happen that $p^t > p^c(\beta)$ for the values of β near 1 (e.g., see **Table 4**).

3.5. Optimality Criterion for β

In order to find an optimal (in some sense) value of the degree of "socialization," β, of the semi-public company, we study the demeanor of the private agent's profit function in two equilibrium states: the consistent CVE (CCVE) and Cournot-Nash equilibrium.

The function $\pi_1(p, q_1)$ given by Eq. (2) is continuously differentiable with respect to p and q_1, while $p^*(\beta)$, $q_1^*(\beta)$, $p^c(\beta)$, $q_1^c(\beta)$ are continuously differentiable with respect to β. Therefore, for the equilibrium states CCVE and Cournot-Nash, we have that the private agent's net profit values,

$$\pi_1^*(\beta) = p^*(\beta)q_1^*(\beta) - \frac{1}{2}a_1 q_1^*(\beta)^2 - b_1 q_1^*(\beta), \quad (23)$$

in the interior equilibrium (CCVE), as well as the similar values

$$\pi_1^c(\beta) = p^c(\beta)q_1^c(\beta) - \frac{1}{2}a_1 q_1^c(\beta)^2 - b_1 q_1^c(\beta), \quad (24)$$

in the Cournot-Nash exterior equilibrium, are continuously differentiable by $\beta \in (0,1]$.

Theorem 8. *The functions $\pi_1^*(\beta)$ and $\pi_1^c(\beta)$ are strictly decreasing with respect to $\beta \in (0,1]$. Moreover, the following inequalities hold:*

$$\pi_1^*(1) > \pi_1^c(1), \quad \ldots\ldots\ldots\ldots \quad (25)$$

and

$$\lim_{\beta \to 0} \pi_1^*(\beta) < \lim_{\beta \to 0} \pi_1^c(\beta). \quad \ldots\ldots\ldots \quad (26)$$

Proof. *See* Appendix C.2.

Table 1. Experiments' input data.

Agent i	b_i	a_i
0	2.0	0.02
1	1.75	0.0175
2	3.25	0.00834

Directly from the proof of the last theorem we conclude that there exists the value $\overline{\beta}$ such that $\pi_1^*(\overline{\beta}) = \pi_1^c(\overline{\beta})$. We now assume that the semi-public firm is socially responsible, and making use of the subsidy policy, it economically motivates the private firm to change its Cournot-Nash strategy to the consistent CVE comportment, or pays subsidies to the consumers to compensate the highest price in the Cournot-Nash equilibrium. The choice of this parameter $\overline{\beta}$ allows the semi-public company not to pay subsidies either to the private company or to the consumers. With this idea in mind, we introduce the following definition:

Definition 3. The value of the parameter $\overline{\beta} \in (0,1)$ such that $\pi_1^*(\overline{\beta}) = \pi_1^c(\overline{\beta})$ is called the *optimal socialization level*.

From Theorem 8, it follows immediately that, for the duopoly model considered in this paper, we can always find the optimal socialization's level for the semi-public company. In other words, the following result has been established above:

Theorem 9. *Under assumptions A1, A2, and A3, there exists the value of $\overline{\beta} \in (0,1)$ such that $\pi_1^*(\overline{\beta}) = \pi_1^c(\overline{\beta})$. In other words, the optimal socialization level definitely exists.*

4. Numerical Results

In this section, we rely on the data of the numerical experiments exposed in the work of Liu et al. [23]. Here, we describe the experiments in more detail.

The inverse demand function is given by

$$p(G,D) = 50 - 0.02(G+D) = 50 - 0.02(q_0 + q_1). \quad (27)$$

Then solving Eq. (27) for $G+D$ yields the demand function

$$G(p) + D = -50p + 2500. \quad \ldots\ldots\ldots \quad (28)$$

The agent's cost functions are quadratic and are described by Eq. (4), where the values a_i and b_i are given in **Table 1**.

We calculate and compare three types of equilibrium: the consistent conjectural variations equilibrium (CCVE), the Cournot-Nash equilibrium, and the perfect competition equilibrium. The influence coefficients for the CCVE are determined by Eqs. (13) and (14). For the Cournot-Nash equilibrium, the influence coefficients are given by the equality $v_i^c = -\frac{\partial p}{\partial q_i} = -\frac{1}{G'(p)} = 0.02$ for all i, while for the perfect competition equilibrium, they have the value

Table 2. Results of Experiment 1.

		$\omega_i = -G'(p)v_i$		
		C-N	**CCVE**	**PC**
$\beta = 0.25$	ω_0	1.0	0.499	0.0
	ω_2	1.0	0.579	0.0
	p	22.609	19.486879	13.595
	q_0	686.425	710.319	579.771
	q_2	683.109	815.337	1240.458
	G	1369.534	1525.656	1820.229
	π_0	14124.01	13194.80	11644.43
	π_2	11278.65	10466.42	6416.53
$\beta = 0.50$	ω_0	1.0	0.493	0.0
	ω_2	1.0	0.555	0.0
	p	20.859	18.324	13.595
	q_0	835.733	808.191	579.771
	q_2	621.335	775.585	1240.458
	G	1457.068	1583.776	1820.229
	π_0	19391.53	19203.30	19927.51
	π_2	9331.00	9183.15	6416.53
$\beta = 0.75$	ω_0	1.0	0.486	0.0
	ω_2	1.0	0.529	0.0
	p	18.869	17.108	13.595
	q_0	1005.431	911.732	579.771
	q_2	551.125	732.863	1240.458
	G	1556.556	1644.595	1820.229
	π_0	25023.08	25747.19	28210.60
	π_2	7341.37	7916.43	6416.53
$\beta = 1.0$	ω_0	1.0	0.478	0.0
	ω_2	1.0	0.500	0.0
	p	16.588	15.846	13.595
	q_0	1200.000	1020.860	579.771
	q_2	470.625	686.823	1240.458
	G	1670.625	1707.683	1820.229
	π_0	31014.88	32875.44	36493.68
	π_2	5353.35	6684.36	6416.53

$v_i^t = -\frac{\partial p}{\partial q_i} = 0$ for any i.

Based on the data of **Table 1**, we proceed to perform the numerical experiments for the following three instances:

Experiment 1: Firm $i = 0$ is semi-public and firm $i = 2$ is private.

Experiment 2: Firm $i = 0$ is semi-public and firm $i = 1$ is private.

Experiment 3: Firm $i = 2$ is semi-public and firm $i = 1$ is private.

In each instance, we handle the following notation for each of the three kinds of equilibrium:

C-N: Cournot-Nash Equilibrium.

CCVE: Consistent Conjectural Variations Equilibrium.

PC: Perfect Competition Equilibrium.

4.1. Experiment 1

For this instance, firm $i = 0$ is semi-public and firm $i = 2$ is private, so that the semi-public firm is stronger than the private firm; that is, the inequality $b_0 \leq b_1$ holds (assumption **A2**). The numerical results of this experiment are shown in **Table 2**.

From the results of **Table 2**, we see that the behavior of variables is as described in the theorems of the previous sections. The numerical results show that, for socialization levels $0 < \beta \leq 0.50$, the private company's profit is higher in the Cournot-Nash equilibrium than in the CCVE, but for $0.75 \leq \beta \leq 1$, its profit is higher in the CCVE equilibrium than in the Cournot-Nash. Then, the optimal socialization level lies within the interval $0.50 < \beta_{\text{optimal}} < 0.75$. Furthermore, as a result of the numerical experiment, the approximate optimal value $\beta_{\text{optimal}} = 0.55262$ is found, for which the private firm's net profit is almost the same both in the Cournot-Nash (C-N) and the Consistent Conjectural Variations Equilibrium (CCVE). The corresponding CCVE (interior equilibrium) is presented as follows: $(p^*, q_0^*, q_1^*, v_0^*, v_1^*) = (18.073, 829.53, 766.82, 0.009830, 0.010991)$. It means that, if the semi-public agent $i = 0$ accepts its objective function as a mixture of 55% of domestic social surplus and 45% of its (would-be) net profit, then the private (foreign) competitor is indifferent to the choice of the Cournot-Nash or CCVE model to generate its supply because its net profit is the same in both cases. This can be considered as a win-win outcome for the local authorities since they needn't either subsidize the consumers (in order to reimburse the higher price of the commodity in the Cournot-Nash equilibrium) or pay a compensation to the private (foreign) firm for having accepted the CCVE equilibrium model (which uses to decrease the private company's net profit as compared to that in the Cournot-Nash equilibrium).

In the previous sections, we made use of assumption **A2**. In the following two experiments, we will consider the case when this assumption is not met to see how our model behaves.

4.2. Experiment 2

In this instance, firm $i = 0$ is semi-public and firm $i = 1$ is private, so that the semi-public firm is weaker than the private firm. The numerical results of this experiment are shown in **Table 3**.

The results shown in **Table 3** demonstrate that the variables still behave as described in the theorems of the previous sections, even though assumption **A2** is not met.

In this second experiment, we have that $\beta_{\text{optimal}} = 0.62905$ and its corresponding interior equilibrium is as follows: $(p^*, q_0^*, q_1^*, v_0^*, v_1^*) = (19.458, 905.37, 621.72, 0.01175, 0.01098)$.

4.3. Experiment 3

For this instance, firm $i = 2$ is semi-public and firm $i = 1$ is private, so that the semi-public firm now is even weaker than the private firm in comparison to the previous experiment. Again, we observe that the variables behave according to our model. The numerical results of this experiment are presented in **Table 4**.

For the third instance, we have that $\beta_{\text{optimal}} = 0.80324$ and its corresponding interior equilibrium is as follows: $(p^*, q_0^*, q_1^*, v_0^*, v_1^*) = (13.329, 1358.7, 474.81, 0.010988, 0.006886)$.

Table 3. Results of Experiment 2.

		$\omega_i = -G'(p)\nu_i$		
		C-N	**CCVE**	**PC**
$\beta = 0.25$	ω_0	1.0	0.594	0.0
	ω_1	1.0	0.591	0.0
	p	23.939	21.564	17.182
	q_0	711.354	746.033	759.091
	q_1	591.703	675.747	881.818
	G	1303.057	1421.781	1640.909
	π_0	14790.94	14083.68	12493.65
	π_1	10065.73	9393.97	6804.03
$\beta = 0.50$	ω_0	1.0	0.590	0.0
	ω_1	1.0	0.564	0.0
	p	22.112	20.203	17.182
	q_0	851.402	848.829	759.091
	q_1	542.991	641.044	881.818
	G	1394.393	1489.873	1640.909
	π_0	19596.32	19344.35	19225.11
	π_1	8476.32	8233.19	6804.03
$\beta = 0.75$	ω_0	1.0	0.585	0.0
	ω_1	1.0	0.534	0.0
	p	20.010	18.734	17.182
	q_0	1012.563	960.608	759.091
	q_1	486.935	602.694	881.818
	G	1499.50	1563.30	1640.91
	π_0	24847.17	25176.45	25956.56
	π_1	6816.78	7057.78	6804.03
$\beta = 1.0$	ω_0	1.0	0.579	0.0
	ω_1	1.0	0.500	0.0
	p	17.565	17.155	17.182
	q_0	1200.000	1082.067	759.091
	q_1	421.739	560.182	881.818
	G	1621.739	1642.249	1640.909
	π_0	30578.64	31659.88	32688.02
	π_1	5113.59	5883.83	6804.03

Table 4. Results of Experiment 3.

		$\omega_i = -G'(p)\nu_i$		
		C-N	**CCVE**	**PC**
$\beta = 0.25$	ω_2	1.0	0.571	0.0
	ω_1	1.0	0.458	0.0
	p	21.529	18.100	13.168
	q_2	896.136	981.770	1189.174
	q_1	527.431	613.227	652.441
	G	1423.567	1594.997	1841.615
	π_2	18097.76	16920.03	14375.80
	π_1	7997.77	6735.88	3724.69
$\beta = 0.50$	ω_2	1.0	0.563	0.0
	ω_1	1.0	0.411	0.0
	p	18.866	16.056	13.168
	q_2	1100.309	1141.023	1189.174
	q_1	456.414	556.192	652.441
	G	1556.723	1697.216	1841.615
	π_2	24250.32	23585.23	22854.66
	π_1	5989.02	5249.90	3724.69
$\beta = 0.75$	ω_2	1.0	0.552	0.0
	ω_1	1.0	0.357	0.0
	p	15.653	13.827	13.168
	q_0	1346.620	1318.492	1189.174
	q_2	370.741	490.173	652.441
	G	1717.361	1808.665	1841.615
	π_2	31259.98	31230.60	31333.53
	π_1	3951.66	3817.31	3724.69
$\beta = 1.0$	ω_2	1.0	0.539	0.0
	ω_1	1.0	0.294	0.0
	p	11.701	11.425	13.168
	q_2	1649.612	1515.019	1189.174
	q_1	265.352	413.722	652.441
	G	1914.964	1928.741	1841.615
	π_2	39263.80	40014.65	39812.39
	π_1	2024.34	2505.13	3724.69

From the results of the numerical experiments, we can see that the weaker semi-public company (as compared to the private company), the closer to 1 its optimal socialization level.

5. Conclusions and Future Research

In this paper, we presented mathematically rigorous proofs of the conjectures (cf., [22]) concerning the behavior of the semi-public and private agents of a mixed duopoly of a homogeneous good. The main difference of this work from the classical duopoly models is in the presence of one producer who maximizes not its net profit, but the convex combination of the latter with domestic social surplus. Moreover, we not only studied the classical Cournot-Nash and perfect competition equilibriums in the model, but also the consistent conjectural variations equilibrium (CCVE) introduced and examined previously by numerous authors.

The paper demonstrated the existence (and in the case of linear, or affine, demand function, the uniqueness) of the CCVE and provided elements of comparative static analysis by evaluating the relationships between the equilibrium price and equilibrium production outputs of both the semi-public and private agents in the aforementioned equilibrium types.

Finally, the role of the convex combination parameter $\beta \in [0,1]$ involved in the definition of the objective function of the semi-public (socially responsible) producer was discussed and investigated. Since this parameter can be considered as reflecting the semi-public company's socialization level, we introduced a criterion to estimate its optimal value – namely, we proposed to admit the value of this parameter as desirable (optimal) if, for this parameter value, the net profits of the private producer under the consistent CVE and the Cournot-Nash equilibrium conditions coincide. It can be reasonable when taking into account that, with such a profit equality, the social responsible authorities need not pay any subsidies either to the private producer (to compensate its financial losses if switching from the Cournot-Nash strategy to the consistent conjectures prevailing in CCVE), or to the consumers (in the case when the equilibrium price under Cournot-

Nash equilibrium turns out to be much higher than it would be in the consistent conjectural variations equilibrium, CCVE). Under the additional assumption about the linear (affine) nature of the model's demand function, the existence of such an optimal parameter value $\bar{\beta} \in (0,1)$ is demonstrated.

However, the linearity of the demand function is a serious restriction. Hence, one of the future research aims is to relax this condition and extend the obtained results to the mixed duopoly with the demand function being not necessarily affine. The next step of our research plan is investigate the role of the socialization level parameter in order to find its optimal value in the mixed oligopoly, wherein more than one private agents compete.

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Appendices

Appendices are available at https://www.fujipress.jp/jaciii/jc/jacii002100071125/.

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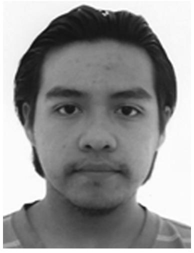

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Appendix A.

A.1. Proof of Theorem 1

Theorem 1. *With postulates **A1**, **A2**, and **A3**, for any $D \geq 0$, $\nu_1 \geq 0$, $i = 0,1$, there exists uniquely the exterior equilibrium $(p^*; q_0^*, q_1^*)$ depending continuously on the parameters $(D; \nu_0, \nu_1)$. The equilibrium price $p^* = p^*(D; \nu_0, \nu_1)$, being a function of those constants, is differentiable by D and ν_i, $i = 0,1$. Furthermore, $p^*(D; \nu_0, \nu_1) > p_0$, and*

$$\frac{\partial p^*}{\partial D} = \frac{1}{\frac{1}{(1-\beta)\nu_0+a_0} + \frac{\nu_0+a_0}{(1-\beta)\nu_0+a_0}\left(\frac{1}{\nu_1+a_1}\right) - G'(p^*)}. \quad (29)$$

Proof. Let $\nu_0, \nu_1 \geq 0$ are fixed. By using the optimality conditions (8) and (10), we can find the output volume functions $q_i = q_i(p; \nu_0, \nu_1)$, $i = 0,1$, defined on the interval $[p_0, +\infty)$. These functions are differentiable with respect to p and ν_i, $i = 0,1$, and they are given by:

$$q_0 = \frac{p-b_0}{(1-\beta)\nu_0+a_0} + \frac{\beta\nu_0}{(1-\beta)\nu_0+a_0}\left(\frac{p-b_1}{\nu_1+a_1}\right), \quad (30)$$

$$q_1 = \frac{p-b_1}{\nu_1+a_1}. \quad (31)$$

Now we introduce the following function:

$$\begin{aligned} Q(p; \nu_0, \nu_1) &= q_0(p; \nu_0, \nu_1) + q_1(p; \nu_0, \nu_1) \\ &= \frac{p-b_0}{(1-\beta)\nu_0+a_0} + \frac{\beta\nu_0}{(1-\beta)\nu_0+a_0}\left(\frac{p-b_1}{\nu_1+a_1}\right) + \frac{p-b_1}{\nu_1+a_1} \\ &= p\left[\frac{1}{(1-\beta)\nu_0+a_0} + \frac{\nu_0+a_0}{(1-\beta)\nu_0+a_0}\left(\frac{1}{\nu_1+a_1}\right)\right] \\ &\quad - \left[\frac{b_0}{(1-\beta)\nu_0+a_0} + \frac{\nu_0+a_0}{(1-\beta)\nu_0+a_0}\left(\frac{b_1}{\nu_1+a_1}\right)\right]. \end{aligned} \quad (32)$$

As we can see from Eq. (32), the function Q is linear in p with positive slope. Therefore, $Q(p; \nu_0, \nu_1)$ strictly increases with respect to p, and tends to $+\infty$ when $p \to +\infty$. By assumption **A3**, one has that for all $\nu_i \geq 0$, $i = 0,1$,

$$\begin{aligned} Q(p_0; \nu_0, \nu_1) &= q_0(p_0; \nu_0, \nu_1) + q_1(p_0; \nu_0, \nu_1) \\ &= \frac{p_0-b_0}{(1-\beta)\nu_0+a_0} \leq \frac{p_0-b_0}{a_0} < G(p_0) \leq G(p_0) + D. \end{aligned} \quad (33)$$

Hence, $Q(p; \nu_0, \nu_1)$ strictly increases with respect to p, the function $G(p)$ is non-increasing by p, and D is constant, so by inequality (33), there exists a unique value $p^* > p_0$ such that

$$Q(p^*; \nu_0, \nu_1) = G(p^*) + D. \quad (34)$$

For this value p^*, using Eqs. (30) and (31), we compute uniquely the equilibrium output volumes $q_i^* = q_i(p^*; \nu_0, \nu_1)$, $i = 0,1$. So we have established the existence and uniqueness of the exterior equilibrium (p^*, q_0^*, q_1^*) for any $D \geq 0$ and $\nu_i \geq 0$, $i = 0,1$.

Now we are going to show that the equilibrium price p^* of the exterior equilibrium is differentiable with respect to the parameters (D, ν_0, ν_1). From Eq. (34) we get the following relationships:

$$Q(p^*; \nu_0, \nu_1) - G(p^*) - D = 0, \quad (35)$$

and we introduce the following function:

$$\begin{aligned} \Gamma(p^*; D, \nu_0, \nu_1) &= Q(p^*; \nu_0, \nu_1) - G(p^*) - D \\ &= p^*\left[\frac{1}{(1-\beta)\nu_0+a_0} + \frac{\nu_0+a_0}{(1-\beta)\nu_0+a_0}\left(\frac{1}{\nu_1+a_1}\right)\right] \\ &\quad - \left[\frac{b_0}{(1-\beta)\nu_0+a_0} + \frac{\nu_0+a_0}{(1-\beta)\nu_0+a_0}\left(\frac{b_1}{\nu_1+a_1}\right)\right] \\ &\quad - G(p^*) - D. \end{aligned} \quad (36)$$

Thus, we can rewrite Eq. (35) as a functional equation

$$\Gamma(p^*; D, \nu_0, \nu_1) = 0 \quad (37)$$

and compute its partial derivative with respect to p^*:

$$\begin{aligned} \frac{\partial \Gamma}{\partial p^*} &= \frac{1}{(1-\beta)\nu_0+a_0} \\ &\quad + \frac{\nu_0+a_0}{(1-\beta)\nu_0+a_0}\left(\frac{1}{\nu_1+a_1}\right) - G'(p^*) \\ &\geq \frac{1}{(1-\beta)\nu_0+a_0} > 0. \end{aligned} \quad (38)$$

From Eq. (38) we can see that the partial derivative of Γ with respect to p^* is positive. Because of that, Implicit Function Theorem implies that the equilibrium price p^* can be considered as a function $p^* = p^*(D, \nu_0, \nu_1)$, which is differentiable with respect to D and ν_i, $i = 0,1$. Moreover, the partial derivative of the price p^* with respect to D can be found from the equation

$$\frac{\partial \Gamma}{\partial p^*}\frac{\partial p^*}{\partial D} + \frac{\partial \Gamma}{\partial D} = 0. \quad (39)$$

The latter leads to

$$\begin{aligned} \frac{\partial p^*}{\partial D} &= -\frac{\frac{\partial \Gamma}{\partial D}}{\frac{\partial \Gamma}{\partial p^*}} \\ &= \frac{1}{\frac{1}{(1-\beta)\nu_0+a_0} + \frac{\nu_0+a_0}{(1-\beta)\nu_0+a_0}\left(\frac{1}{\nu_1+a_1}\right) - G'(p^*)}. \end{aligned} \quad (40)$$

Finally, since the function p^* depends uon (D, ν_0, ν_1) and is differentiable with respect to D and ν_i, $i = 0,1$, the functions q_i^*, $i = 0,1$, also depend on (D, ν_0, ν_1) and are differentiable with respect to D and ν_i, $i = 0,1$. Therefore, the equilibrium (p^*, q_0^*, q_1^*) continuously depends on the parameters (D, ν_0, ν_1). The proof of the theorem is complete.

A.2. Proof of Theorem 3

Theorem 3. *For all τ there exists a unique solution $\nu_i = \nu_i(\tau)$, $i = 0,1$, of the system* (15) *and* (16)*, which continuously depends on τ. In addition, $\nu_i(\tau) \to 0$ whenever $\tau \to -\infty$, and $\nu_i(\tau)$ strictly grows and tends to $\nu_i(0)$ as $\tau \to 0$, $i = 0,1$.*

Proof. The variables ν_i, $i = 0,1$, given by Eqs. (15) and (16) are considered on their domains: ($\nu_i \geq 0, a_i > 0$, $i = 0,1$, $\beta \in (0,1]$ and $\tau \in (-\infty, 0]$).

Substituting Eq. (16) in Eq. (15) we get the following equation:

$$\begin{aligned} v_0 &= \frac{1}{\frac{1}{\left(\frac{1}{\frac{1}{(1-\beta)v_0+a_0}-\tau}\right)+a_1}-\tau} \\ &= \frac{(1-\beta)(1-a_1\tau)v_0+(a_0+a_1-a_0a_1\tau)}{(1-\beta)\left(-2\tau+a_1\tau^2\right)v_0+\left(1-2a_0\tau-a_1\tau+a_0a_1\tau^2\right)}. \end{aligned} \quad (41)$$

Then, we can multiply Eq. (41) by $\left[(1-\beta)\left(-2\tau+a_1\tau^2\right)v_0+\left(1-2a_0\tau-a_1\tau+a_0a_1\tau^2\right)\right]$ to obtain

$$\begin{aligned} &\left[(1-\beta)\left(-2\tau+a_1\tau^2\right)v_0+\left(1-2a_0\tau-a_1\tau+a_0a_1\tau^2\right)\right]v_0 \\ &\qquad =(1-\beta)(1-a_1\tau)v_0+(a_0+a_1-a_0a_1\tau). \end{aligned} \quad (42)$$

Move all the terms of Eq. (42) to the left-hand side and get

$$\begin{aligned} &\left[(1-\beta)\left(-2\tau+a_1\tau^2\right)v_0+\left(1-2a_0\tau-a_1\tau+a_0a_1\tau^2\right)\right]v_0 \\ &\qquad -(1-\beta)(1-a_1\tau)v_0-(a_0+a_1-a_0a_1\tau)=0. \end{aligned} \quad (43)$$

By extracting a common factors in Eq. (43) we obtain the following quadratic equation for v_0:

$$\begin{aligned} &(1-\beta)\left(-2\tau+a_1\tau^2\right)v_0^2+\left(\beta-2a_0\tau-\beta a_1\tau+a_0a_1\tau^2\right)v_0 \\ &\qquad -(a_0+a_1-a_0a_1\tau)=0. \end{aligned} \quad (44)$$

Now, in order to simplify the notation, we rewrite Eq. (44) as follows:

$$Av_0^2+Bv_0-C=0, \quad (45)$$

where

$$A=A(\tau)=(1-\beta)\left(-2\tau+a_1\tau^2\right)\geq 0, \quad (46)$$

$$B=B(\tau)=\beta-2a_0\tau-\beta a_1\tau+a_0a_1\tau^2>0, \quad (47)$$

$$C=C(\tau)=a_0+a_1-a_0a_1\tau>0. \quad (48)$$

If $\tau=0$ or $\beta=1$, then, $A=0$ and Eq. (45) is linear, so we can find the unique solution for v_0 given by:

$$v_0(\tau)=\begin{cases} \frac{a_0+a_1}{\beta} & \text{if } \tau=0 \\ \frac{a_0+a_1-a_0a_1\tau}{1-2a_0\tau-a_1\tau+a_0a_1\tau^2} & \text{if } \beta=1 \end{cases}. \quad (49)$$

If $\beta\in(0,1)$ and $\tau<0$, then, $A\neq 0$ and we can find both roots of Eq. (45), which are:

$$v_0(\tau)=\frac{-B+\sqrt{B^2+4AC}}{2A}, \quad (50)$$

$$v_0(\tau)=\frac{-B-\sqrt{B^2+4AC}}{2A}. \quad (51)$$

However, since $v_0\geq 0$, the root (51) is impossible; that is, Eq. (50) is the unique solution of Eq. (45).

Moreover, Eqs. (49) and (50) can be combined in a single equation for all $\beta\in(0,1]$ and $\tau\in(-\infty,0]$ as follows:

$$v_0(\tau)=v_0=\frac{2C}{B+\sqrt{B^2+4AC}}, \quad (52)$$

where

$$B+\sqrt{B^2+4AC}>0, \quad (53)$$

and so Eq. (52) is the unique solution for v_0.

We can see that the solution (52) for any parameter $\beta\in(0,1]$ satisfies the condition $v_0\to 0$ as $\tau\to-\infty$. Hence, there exits a positive value $\overline{v_0}(\beta)$ such that $v_0(\tau)\leq\overline{v_0}(\beta)$ for all $\tau\leq 0$.

From Eqs. (16) and (52), we can see that v_1 also has a unique solution, which is given by

$$v_1(\tau)=v_1=\frac{1}{\frac{1}{(1-\beta)v_0(\tau)+a_0}-\tau}. \quad (54)$$

For any parameter $\beta\in(0,1]$, the conditions $v_1\to 0$ as $\tau\to-\infty$ and $v_1(\tau)\leq a_0+(1-\beta)\overline{v_0}(\beta)$ for all $\tau\leq 0$, are satisfied.

Now, it is apparent that the functions (46)–(48) are continuously differentiable with respect to τ, $\tau\in(-\infty,0]$ and

$$A'=(1-\beta)(-2+2a_1\tau)\leq 0, \quad (55)$$

$$B'=-2a_0-\beta a_1+2a_0a_1\tau<0, \quad (56)$$

$$C'=-a_0a_1<0. \quad (57)$$

Thus, from Eq. (52), we have that $v_0(\tau)$ is continuously differentiable and

$$\begin{aligned} v_0' &= \frac{2C'\left(B+\sqrt{B^2+4AC}\right)-2C\left(B'+\frac{2BB'+4A'C+4AC'}{2\sqrt{B^2+4AC}}\right)}{\left(B+\sqrt{B^2+4AC}\right)^2} \\ &= \frac{2\left(C'B-CB'\right)\left(B+\sqrt{B^2+4AC}\right)+4\left(AC'-A'C\right)C}{\left(B+\sqrt{B^2+4AC}\right)^2\sqrt{B^2+4AC}}. \end{aligned} \quad (58)$$

Now, we estimate the values of Eq. (58) in order to reveal the behavior of $v_0(\tau)$ as the function of τ.

From Eqs. (46)–(48), it is evident that the denominator of Eq. (58) is positive:

$$\left(B+\sqrt{B^2+4AC}\right)^2\sqrt{B^2+4AC}>0. \quad (59)$$

Thus, plugging Eqs. (46)–(48) and (55)–(57) in Eq. (58), we can find that

$$\begin{aligned} C'B-CB' &= (-a_0a_1)\left(\beta-2a_0\tau-\beta a_1\tau+a_0a_1\tau^2\right) \\ &\quad -(a_0+a_1-a_0a_1\tau)(-2a_0-\beta a_1+2a_0a_1\tau) \\ &= \beta a_1^2+a_0^2a_1^2\tau^2+(a_0+a_1)(2a_0-2a_0a_1\tau)>0, \end{aligned} \quad (60)$$

and

$$\begin{aligned} AC'-A'C &= (1-\beta)\left(-2\tau+a_1\tau^2\right)(-a_0a_1) \\ &\quad -(1-\beta)(-2+2a_1\tau)(a_0+a_1-a_0a_1\tau) \\ &= (1-\beta)a_0a_1^2\tau^2+(1-\beta)(2-2a_1\tau)(a_0+a_1)\geq 0. \end{aligned} \quad (61)$$

Therefore, given the values of Eqs. (46)–(48), (53), and (59)–(61), we can conclude that

$$\begin{aligned} v_0' &= \frac{2\left(C'B-CB'\right)\left(B+\sqrt{B^2+4AC}\right)+4\left(AC'-A'C\right)C}{\left(B+\sqrt{B^2+4AC}\right)^2\sqrt{B^2+4AC}} \\ &\geq \frac{2\left(C'B-CB'\right)\left(B+\sqrt{B^2+4AC}\right)}{\left(B+\sqrt{B^2+4AC}\right)^2\sqrt{B^2+4AC}}>0. \end{aligned} \quad (62)$$

Therefore, $v_0(\tau)$ is strictly increasing with respect to $\tau, \tau\in(-\infty,0]$. Since the function $v_0=v_0(\tau)$ is continuous, it tends to $v_0(0)$ as τ goes to 0.

Now, from Eq. (54) we have

$$v_1 = \frac{1}{\frac{1}{(1-\beta)v_0+a_0} - \tau}. \quad (63)$$

Since $v_0(\tau)$ is continuously differentiable with respect to τ, the same is true for $v_1(\tau)$, and

$$\begin{aligned} v_1' &= -\frac{1}{\left(\frac{1}{(1-\beta)v_0+a_0} - \tau\right)^2}\left(-\frac{1}{[(1-\beta)v_0+a_0]^2}(1-\beta)v_0' - 1\right) \\ &= v_1^2\left(\frac{(1-\beta)v_0'}{[(1-\beta)v_0+a_0]^2}+1\right), \end{aligned} \quad (64)$$

where $v_0' > 0$. On account of that,

$$v_1' = v_1^2\left(\frac{(1-\beta)v_0'}{[(1-\beta)v_0+a_0]^2}+1\right) \geq v_1^2 > 0. \quad (65)$$

Therefore, $v_1(\tau)$ strictly increases with respect to $\tau, \tau \in (-\infty, 0]$. Since the function $v_1 = v_1(\tau)$ is continuous, it tends to $v_1(0)$ as τ goes to 0.The proof of the theorem is complete.

A.3. Proof of Theorem 2

Theorem 2. *If assumptions **A1**, **A2**, and **A3** hold, there exist interior equilibria.*

Proof. We are going to show that there exist $v_i^* \geq 0, q_i^* \geq 0$, $i = 0,1$, and $p^* > p_0$ such that the vector (p^*, q_0^*, q_1^*) is the exterior equilibrium, and the influence coefficients (v_0^*, v_1^*) are consistent, i.e., Eqs. (13) and (14) hold.

As it was proved in **Theorem 3**, v_0 and v_1 solve uniquely Eqs. (15) and (16), and continuously depend on $\tau = G'(p)$. Moreover, $G'(p)$ continuously depends on p, hence, the functions v_0 and v_1 are continuous with respect to p.

Recall the function (32) introduced when proving **Theorem 1**:

$$\begin{aligned} &Q(p; v_0(p), v_1(p)) \\ &= \frac{p-b_0}{(1-\beta)v_0(p)+a_0} + \frac{p-b_1}{v_1(p)+a_1} + \frac{\beta v_0(p)}{(1-\beta)v_0(p)+a_0}\left(\frac{p-b_1}{v_1(p)+a_1}\right) \\ &= p\left[\frac{1}{(1-\beta)v_0(p)+a_0} + \frac{v_0(p)+a_0}{(1-\beta)v_0(p)+a_0}\left(\frac{1}{v_1(p)+a_1}\right)\right] \\ &\quad - \left[\frac{b_0}{(1-\beta)v_0(p)+a_0} + \frac{v_0(p)+a_0}{(1-\beta)v_0(p)+a_0}\left(\frac{b_1}{v_1(p)+a_1}\right)\right], \end{aligned} \quad (66)$$

which continuously depends on p and tends to $+\infty$ as $p \to +\infty$ since $v_0(p)$ and $v_1(p)$ are bounded. Thus, by assumption **A3**, we have that

$$\begin{aligned} Q(p_0) = \frac{p_0-b_0}{(1-\beta)v_0(p_0)+a_0} &\leq \frac{p_0-b_0}{a_0} \\ &< G(p_0) \leq G(p_0)+D. \end{aligned} \quad (67)$$

Therefore, there exists the value $p^* > p_0$ such that

$$Q(p^*) = G(p^*)+D. \quad (68)$$

For this value p^*, we compute the influence coefficients $v_i^* = v_i(G'(p^*))$, $i = 0,1$, using Eqs. (52) and (54), as well as the output volumes $q_i^* = q_i(p^*; v_0^*, v_1^*)$, $i = 0,1$, given by Eqs. (30) and (31). Thus, v_0^* and v_1^* satisfy Eqs. (13) and (14), whereas the vector (p^*, q_0^*, q_1^*) is the exterior equilibrium. As a consequence, the extended vector $(p^*, q_0^*, q_1^*, v_0^*, v_1^*)$ is the interior equilibrium. The proof of the theorem is complete.

A.4. Proof of Corollary 1

Corollary 1. *When conditions **A1**, **A2**, and **A3** are valid, then the demand function of type* (17) *and all $\beta \in (0,1]$ imply the existence of the (unique) interior equilibrium.*

Proof. Consider an arbitrary $\beta \in (0,1]$. Since $G'(p) = -K$, then, by **Theorem 3**, for $\tau = -K$ there exists a unique solution (v_0^*, v_1^*) of Eqs. (15) and (16):

$$v_0^* = \frac{\alpha_0}{\alpha_1}, \quad (69)$$

where

$$\alpha_0 = 2(a_0 + a_1 + a_0 a_1 K) \quad (70)$$

and

$$\begin{aligned} \alpha_1 &= \left(\beta + 2a_0K + \beta a_1 K + a_0 a_1 K^2\right) \\ &+ \sqrt{\left(\beta+2a_0K+\beta a_1K+a_0a_1K^2\right)^2+4(1-\beta)\left(2K+a_1K^2\right)(a_0+a_1+a_0a_1K)}. \end{aligned} \quad (71)$$

In addition,

$$v_1^* = \frac{1}{\frac{1}{(1-\beta)v_0^*+a_0}+K} = \frac{(1-\beta)v_0^*+a_0}{1+\left[(1-\beta)v_0^*+a_0\right]K}. \quad (72)$$

Moreover, from Eq. (15), we can rewrite Eq. (69) as follows:

$$v_0^* = \frac{1}{\frac{1}{v_1^*+a_1}+K}. \quad (73)$$

It is not difficult to see that the influence coefficients v_0^* and v_1^* don't depend on p, therefore, by **Theorem 1**, there exists the unique exterior equilibrium (p^*, q_0^*, q_1^*) with the influence coefficients (v_0^*, v_1^*). Hence, the vector $(p^*, q_0^*, q_1^*, v_0^*, v_1^*) = (p^*(\beta), q_0^*(\beta), q_1^*(\beta), v_0^*(\beta), v_1^*(\beta))$ is the unique interior equilibrium for $\beta \in (0,1]$. The proof of the corollary is complete.

Appendix B.

B.1. Proof of Theorem 4

Theorem 4. *For the linear (affine) demand function $G(p)$ from Eq.* (17), *the price $p^*(\beta)$, the outputs $q_i^*(\beta)$, $i = 0,1$, and the influence coefficients $v_i^*(\beta)$, $i = 0,1$, characterizing the (unique) interior equilibrium, together with the total market supply $G^*(\beta) = q_0^*(\beta) + q_1^*(\beta)$, are continuously differentiable by $\beta \in (0,1]$. Furthermore, $q_0^*(\beta)$ and $G^*(\beta)$ are strictly growing whereas $p^*(\beta), v_0^*(\beta), v_1^*(\beta)$, and $q_1^*(\beta)$ strictly decrease.*

Proof. First, we are going to show that the functions $v_i^*(\beta), i = 0,1$, are continuously differentiable and strictly decreasing with respect to β. Let us consider the functions

$$\mathscr{A}=\mathscr{A}(\beta)=(1-\beta)\left(2K+a_1K^2\right)\geq 0, \quad (74)$$

$$\mathscr{B}=\mathscr{B}(\beta)=\beta+2a_0K+\beta a_1K+a_0a_1K^2>0, \quad (75)$$

$$\mathscr{C}=\mathscr{C}(\beta)=a_0+a_1+a_0a_1K>0, \quad (76)$$

which are continuously differentiable with respect to β, with

$$\mathscr{A}'=-\left(2K+a_1K^2\right)<0, \quad (77)$$

$$\mathscr{B}'=1+a_1K>0, \quad (78)$$

$$\mathscr{C}'=0. \quad (79)$$

Using Eqs. (74)–(76) we rewrite Eq. (69) as follows:

$$v_0^*(\beta)=v_0^*=\frac{2\mathscr{C}}{\mathscr{B}+\sqrt{\mathscr{B}^2+4\mathscr{A}\mathscr{C}}}, \quad (80)$$

where

$$\mathscr{B}+\sqrt{\mathscr{B}^2+4\mathscr{A}\mathscr{C}}>0. \quad (81)$$

Then, $v_0^*(\beta)$ is continuously differentiable with respect to β and, similarly to Eq. (58),

$$v_0^{*\prime}=\frac{2\left(\mathscr{C}'\mathscr{B}-\mathscr{C}\mathscr{B}'\right)\left(\mathscr{B}+\sqrt{\mathscr{B}^2+4\mathscr{A}\mathscr{C}}\right)+4\left(\mathscr{A}\mathscr{C}'-\mathscr{A}'\mathscr{C}\right)\mathscr{C}}{\left(\mathscr{B}+\sqrt{\mathscr{B}^2+4\mathscr{A}\mathscr{C}}\right)^2\sqrt{\mathscr{B}^2+4\mathscr{A}\mathscr{C}}}. \quad (82)$$

Since $\mathscr{C}'=0$, then

$$\begin{aligned}v_0^{*\prime}&=\frac{2\left(-\mathscr{C}\mathscr{B}'\right)\left(\mathscr{B}+\sqrt{\mathscr{B}^2+4\mathscr{A}\mathscr{C}}\right)+4\left(-\mathscr{A}'\mathscr{C}\right)\mathscr{C}}{\left(\mathscr{B}+\sqrt{\mathscr{B}^2+4\mathscr{A}\mathscr{C}}\right)^2\sqrt{\mathscr{B}^2+4\mathscr{A}\mathscr{C}}}\\&=\frac{-2\mathscr{C}\left[\mathscr{B}'\left(\mathscr{B}+\sqrt{\mathscr{B}^2+4\mathscr{A}\mathscr{C}}\right)+2\mathscr{A}'\mathscr{C}\right]}{\left(\mathscr{B}+\sqrt{\mathscr{B}^2+4\mathscr{A}\mathscr{C}}\right)^2\sqrt{\mathscr{B}^2+4\mathscr{A}\mathscr{C}}}.\end{aligned} \quad (83)$$

Now we are going to estimate the value of Eq. (83) in order to describe the behavior of $v_0^*(\beta)$ as a function of β.

From Eqs. (74)–(76), it is evident that the denominator of Eq. (83) is positive:

$$\left(\mathscr{B}+\sqrt{\mathscr{B}^2+4\mathscr{A}\mathscr{C}}\right)^2\sqrt{\mathscr{B}^2+4\mathscr{A}\mathscr{C}}>0. \quad (84)$$

Suppose that the numerator of Eq. (83) is non-negative for some $\beta_0\in(0,1]$, i.e.,

$$-2\mathscr{C}\left[\mathscr{B}'\left(\mathscr{B}+\sqrt{\mathscr{B}^2+4\mathscr{A}\mathscr{C}}\right)+2\mathscr{A}'\mathscr{C}\right]\geq 0. \quad (85)$$

Since $\mathscr{C}>0$, by Eq. (76), we have that

$$\mathscr{B}'\left(\mathscr{B}+\sqrt{\mathscr{B}^2+4\mathscr{A}\mathscr{C}}\right)+2\mathscr{A}'\mathscr{C}\leq 0. \quad (86)$$

Moreover, $B'>0$, by Eq. (78), therefore,

$$\sqrt{\mathscr{B}^2+4\mathscr{A}\mathscr{C}}\leq\frac{-2\mathscr{A}'\mathscr{C}}{\mathscr{B}'}-\mathscr{B}, \quad (87)$$

where $\sqrt{\mathscr{B}^2+4\mathscr{A}\mathscr{C}}>0$. Now squaring both sides of Eq. (87) we have

$$\mathscr{B}^2+4\mathscr{A}\mathscr{C}\leq\frac{4\mathscr{A}'^2\mathscr{C}^2}{\mathscr{B}'^2}+\frac{4\mathscr{A}'\mathscr{C}\mathscr{B}}{\mathscr{B}'}+\mathscr{B}^2. \quad (88)$$

Solving Eq. (88) for $\mathscr{A}$ we get

$$\mathscr{A}\leq\frac{\mathscr{A}'^2\mathscr{C}}{\mathscr{B}'^2}+\frac{\mathscr{A}'\mathscr{B}}{\mathscr{B}'}. \quad (89)$$

Multiplying both sides of Eq. (89) by $\mathscr{B}'^2$ we deduce

$$\mathscr{A}\mathscr{B}'^2\leq\mathscr{A}'^2\mathscr{C}+\mathscr{A}'\mathscr{B}\mathscr{B}'=\mathscr{A}'\left(\mathscr{A}'\mathscr{C}+\mathscr{B}\mathscr{B}'\right). \quad (90)$$

Now, we substitute the values of A and A' given by Eqs. (74) and (77) in Eq. (90) to obtain:

$$\begin{aligned}&(1-\beta)\left(2K+a_1K^2\right)\mathscr{B}'^2\\&\leq-\left(2K+a_1K^2\right)\left[-\left(2K+a_1K^2\right)\mathscr{C}+\mathscr{B}\mathscr{B}'\right],\end{aligned} \quad (91)$$

and since $\left(2K+a_1K^2\right)>0$, we have that

$$\begin{aligned}&(1-\beta)\mathscr{B}'^2\leq-\left[-\left(2K+a_1K^2\right)\mathscr{C}+\mathscr{B}\mathscr{B}'\right]\\&=\left(2K+a_1K^2\right)\mathscr{C}-\mathscr{B}\mathscr{B}'.\end{aligned} \quad (92)$$

The latter implies

$$\begin{aligned}&(1-\beta)\mathscr{B}'^2+\mathscr{B}\mathscr{B}'-\left(2K+a_1K^2\right)\mathscr{C}\\&=\left[(1-\beta)\mathscr{B}'+\mathscr{B}\right]\mathscr{B}'-\left(2K+a_1K^2\right)\mathscr{C}\leq 0.\end{aligned} \quad (93)$$

Plugging Eqs. (75), (76) and (78) in Eq. (93) we yield

$$\begin{aligned}&\left[(1-\beta)\mathscr{B}'+\mathscr{B}\right]\mathscr{B}'-\left(2K+a_1K^2\right)\mathscr{C}\\&=\left[(1-\beta)(1+a_1K)+\right.\\&\quad\left.+\left(\beta+2a_0K+\beta a_1K+a_0a_1K^2\right)\right](1+a_1K)\\&\quad-\left(2K+a_1K^2\right)(a_0+a_1+a_0a_1K)\\&=1>0,\end{aligned} \quad (94)$$

which contradicts Eq. (93). Hence, Eq. (85) cannot hold for any $\beta_0\in(0,1]$, which implies

$$-2\mathscr{C}\left[\mathscr{B}'\left(\mathscr{B}+\sqrt{\mathscr{B}^2+4\mathscr{A}\mathscr{C}}\right)+2\mathscr{A}'\mathscr{C}\right]<0 \quad (95)$$

for all $\beta\in(0,1]$.

Therefore, from Eqs. (84) and (95), we conclude that

$$v_0^{*\prime}=\frac{-2\mathscr{C}\left[\mathscr{B}'\left(\mathscr{B}+\sqrt{\mathscr{B}^2+4\mathscr{A}\mathscr{C}}\right)+2\mathscr{A}'\mathscr{C}\right]}{\left(\mathscr{B}+\sqrt{\mathscr{B}^2+4\mathscr{A}\mathscr{C}}\right)^2\sqrt{\mathscr{B}^2+4\mathscr{A}\mathscr{C}}}<0 \quad (96)$$

for all $\beta\in(0,1]$. On account of the latter, $v_0^*(\beta)$ is continuously differentiable and strictly decreasing with respect to $\beta\in(0,1]$.

From Eq. (72), it is transparent that v_1^* is continuously differentiable with respect to v_0^* and, since $v_0^*(\beta)$, in its turn, is also smooth as a function of β, then $v_1^*(\beta)$ is continuously differentiable by β.

Differentiating Eq. (73) with respect to β we get

$$\begin{aligned}v_0^{*\prime}&=\frac{1}{\left(\frac{1}{v_1^*+a_1}+K\right)^2}\left(\frac{1}{\left(v_1^*+a_1\right)^2}v_1^{*\prime}\right)\\&=v_0^{*2}\left(\frac{v_1^{*\prime}}{\left(v_1^*+a_1\right)^2}\right)=\left(\frac{v_0^*}{v_1^*+a_1}\right)^2v_1^{*\prime}.\end{aligned} \quad (97)$$

Since $v_0^{*\prime} < 0$, then Eq. (97) implies that $v_1^{*\prime} < 0$, for all $\beta \in (0,1]$. Thus, $v_1^*(\beta)$ is continuously differentiable and strictly decreasing with respect to β, $\beta \in (0,1]$. Before continuing the proof, we are going to establish the following inequality:

$$v_0^* + \beta v_0^{*\prime} > 0. \quad (98)$$

Substituting Eqs. (80) and (83) in Eq. (98) we get

$$\begin{aligned} v_0^* + \beta v_0^{*\prime} &= \frac{2\mathscr{C}}{\mathscr{B}+\sqrt{\mathscr{B}^2+4\mathscr{A}\mathscr{C}}} \\ &\quad + \beta \frac{-2\mathscr{C}\left[\mathscr{B}'\left(\mathscr{B}+\sqrt{\mathscr{B}^2+4\mathscr{A}\mathscr{C}}\right)+2\mathscr{A}'\mathscr{C}\right]}{\left(\mathscr{B}+\sqrt{\mathscr{B}^2+4\mathscr{A}\mathscr{C}}\right)^2\sqrt{\mathscr{B}^2+4\mathscr{A}\mathscr{C}}} \\ &= \frac{2\mathscr{C}}{\left(\mathscr{B}+\sqrt{\mathscr{B}^2+4\mathscr{A}\mathscr{C}}\right)^2\sqrt{\mathscr{B}^2+4\mathscr{A}\mathscr{C}}} \\ &\quad \cdot\left[\left(-\beta\mathscr{B}'+\sqrt{\mathscr{B}^2+4\mathscr{A}\mathscr{C}}\right)\left(\mathscr{B}+\sqrt{\mathscr{B}^2+4\mathscr{A}\mathscr{C}}\right)-2\beta\mathscr{A}'\mathscr{C}\right]. \end{aligned} \quad (99)$$

By Eqs. (76), (77), and (84),

$$\frac{2\mathscr{C}}{\left(\mathscr{B}+\sqrt{\mathscr{B}^2+4\mathscr{A}\mathscr{C}}\right)^2\sqrt{\mathscr{B}^2+4\mathscr{A}\mathscr{C}}} > 0 \quad (100)$$

and

$$-2\beta\mathscr{A}'\mathscr{C} > 0. \quad (101)$$

Then, to prove inequality (98), it suffices to show that

$$\left(-\beta\mathscr{B}'+\sqrt{\mathscr{B}^2+4\mathscr{A}\mathscr{C}}\right)\left(\mathscr{B}+\sqrt{\mathscr{B}^2+4\mathscr{A}\mathscr{C}}\right)>0, \quad (102)$$

which, by Eq. (81), is equivalent to demonstrating that

$$-\beta\mathscr{B}'+\sqrt{\mathscr{B}^2+4\mathscr{A}\mathscr{C}} > 0. \quad (103)$$

Suppose, on the contrary, that

$$-\beta\mathscr{B}'+\sqrt{\mathscr{B}^2+4\mathscr{A}\mathscr{C}} \le 0. \quad (104)$$

Then

$$\sqrt{\mathscr{B}^2+4\mathscr{A}\mathscr{C}} \le \beta\mathscr{B}', \quad (105)$$

where $\sqrt{\mathscr{B}^2+4\mathscr{A}\mathscr{C}} > 0$. Hence, by squaring both sides of Eq. (105) we have

$$\mathscr{B}^2+4\mathscr{A}\mathscr{C} \le \beta^2\mathscr{B}'^2. \quad (106)$$

Plugging Eqs. (75) and (78) in Eq. (106) yields

$$\left(\beta+2a_0K+\beta a_1K+a_0a_1K^2\right)^2+4\mathscr{A}\mathscr{C}\le\beta^2(1+a_1K)^2, \quad (107)$$

which implies

$$\left[(\beta+\beta a_1K)+2a_0K+a_0a_1K^2\right]^2+4\mathscr{A}\mathscr{C}\le(\beta+\beta a_1K)^2. \quad (108)$$

However, by Eqs. (74) and (76),

$$4\mathscr{A}\mathscr{C} \ge 0, \quad (109)$$

that is,

$$\left[(\beta+\beta a_1K)+2a_0K+a_0a_1K^2\right]^2\le(\beta+\beta a_1K)^2. \quad (110)$$

On the other hand,

$$2a_0K + a_0a_1K^2 > 0, \quad (111)$$

whence

$$(\beta+\beta a_1K)<(\beta+\beta a_1K)+2a_0K+a_0a_1K^2, \quad (112)$$

where $(\beta+\beta a_1K) > 0$. Now by squaring both sides of Eq. (105) we have

$$(\beta+\beta a_1K)^2<\left[(\beta+\beta a_1K)+2a_0K+a_0a_1K^2\right]^2. \quad (113)$$

Nevertheless, inequality (113) contradicts Eq. (110), which means that Eq. (103) must hold and thus prove Eq. (98).

Now, coming back to the proof of the theorem, we are going to show that the equilibrium price $p^*(\beta)$ is continuously differentiable and strictly decreasing with respect to β. Consider again the function (32) and by plugging it in $G(p^*) = -Kp^* + T$ get the following relationships:

$$\begin{aligned} &Q(p^*;v_0^*,v_1^*) - G(p^*) - D \\ &= p^*\left[\frac{1}{(1-\beta)v_0^*+a_0} + \frac{v_0^*+a_0}{(1-\beta)v_0^*+a_0}\left(\frac{1}{v_1^*+a_1}\right)\right] \\ &\quad - \left[\frac{b_0}{(1-\beta)v_0^*+a_0} + \frac{v_0^*+a_0}{(1-\beta)v_0^*+a_0}\left(\frac{b_1}{v_1^*+a_1}\right)\right] \\ &\quad + Kp^* - T - D = 0. \end{aligned} \quad (114)$$

Consider the function

$$\begin{aligned} &\mathscr{F}(p^*;\beta) \\ &= p^*\left[\frac{1}{(1-\beta)v_0^*+a_0} + \frac{v_0^*+a_0}{(1-\beta)v_0^*+a_0}\left(\frac{1}{v_1^*+a_1}\right)\right] \\ &\quad - \left[\frac{b_0}{(1-\beta)v_0^*+a_0} + \frac{v_0^*+a_0}{(1-\beta)v_0^*+a_0}\left(\frac{b_1}{v_1^*+a_1}\right)\right] \\ &\quad + Kp^* - T - D, \end{aligned} \quad (115)$$

having in mind that v_0^* and v_1^* depend on β, but not on p^*. Now, we rewrite Eq. (114) using Eq. (115) as a functional equation:

$$\mathscr{F}(p^*;\beta) = 0. \quad (116)$$

Now we are in a position to estimate the value of the partial derivative of the function $\mathscr{F}(p^*;\beta)$ with respect to p^*:

$$\begin{aligned} \frac{\partial\mathscr{F}}{\partial p^*} &= \frac{1}{(1-\beta)v_0^*+a_0} \\ &\quad + \frac{v_0^*+a_0}{(1-\beta)v_0^*+a_0}\left(\frac{1}{v_1^*+a_1}\right)+K\ge K>0. \end{aligned} \quad (117)$$

We observe that the partial derivative $\mathscr{F}$ with respect to p^* is positive. Hence, by the Implicit Function Theorem, the function $p^* = p^*(\beta)$ is differentiable with respect to β, and its partial derivative with respect to β can be found from the equation

$$\frac{\partial\mathscr{F}}{\partial p^*}\frac{dp^*}{d\beta} + \frac{\partial\mathscr{F}}{\partial\beta} = 0, \quad (118)$$

which leads to

$$\frac{dp^*}{d\beta} = -\frac{\frac{\partial\mathscr{F}}{\partial\beta}}{\frac{\partial\mathscr{F}}{\partial p^*}}. \quad (119)$$

From Eq. (117), we have

$$\frac{\partial\mathscr{F}}{\partial p^*} > 0. \quad (120)$$

Therefore, to prove that p^* is strictly increasing, we have to show that

$$\frac{\partial \mathscr{F}}{\partial \beta} > 0. \quad \dots \quad (121)$$

Indeed,

$$\begin{aligned}
\frac{\partial \mathscr{F}}{\partial \beta} &= \frac{\partial}{\partial \beta} Q(p^*; v_0^*, v_1^*) \\
&= \frac{\partial}{\partial \beta}\left[\frac{p^*-b_0}{(1-\beta)v_0^*+a_0} + \frac{\beta v_0^*}{(1-\beta)v_0^*+a_0}\left(\frac{p^*-b_1}{v_1^*+a_1}\right) + \frac{p^*-b_1}{v_1^*+a_1}\right] \\
&= \frac{p^*-b_0}{\left[(1-\beta)v_0^*+a_0\right]^2}\left[v_0^*+(1-\beta)\left(-v_0^{*\prime}\right)\right] \\
&\quad + \frac{\left(v_0^*+\beta v_0^{*\prime}\right)\left[(1-\beta)v_0^*+a_0\right]+\beta v_0^*\left[v_0^*+(1-\beta)\left(-v_0^{*\prime}\right)\right]}{\left[(1-\beta)v_0^*+a_0\right]^2}\left(\frac{p^*-b_1}{v_1^*+a_1}\right) \\
&\quad + \frac{\beta v_0^*}{(1-\beta)v_0^*+a_0}\left[\frac{p^*-b_1}{\left(v_1^*+a_1\right)^2}\left(-v_1^{*\prime}\right)\right] + \left[\frac{p^*-b_1}{\left(v_1^*+a_1\right)^2}\left(-v_1^{*\prime}\right)\right].
\end{aligned}$$

$$\dots \quad (122)$$

Given the values of $a_0, a_1, b_0, b_1, \beta, v_0^*, v_1^*, v_0^{*\prime}, v_1^{*\prime}, p^*$, and Eq. (98), it isn't difficult to see that Eq. (122) is non-negative. Moreover,

$$\begin{aligned}
\frac{\partial \mathscr{F}}{\partial \beta} &= \frac{p^*-b_0}{\left[(1-\beta)v_0^*+a_0\right]^2}\left[v_0^*+(1-\beta)\left(-v_0^{*\prime}\right)\right] \\
&\quad + \frac{\left(v_0^*+\beta v_0^{*\prime}\right)\left[(1-\beta)v_0^*+a_0\right]+\beta v_0^*\left[v_0^*+(1-\beta)\left(-v_0^{*\prime}\right)\right]}{\left[(1-\beta)v_0^*+a_0\right]^2}\left(\frac{p^*-b_1}{v_1^*+a_1}\right) \\
&\quad + \frac{\beta v_0^*}{(1-\beta)v_0^*+a_0}\left[\frac{p^*-b_1}{\left(v_1^*+a_1\right)^2}\left(-v_1^{*\prime}\right)\right] + \left[\frac{p^*-b_1}{\left(v_1^*+a_1\right)^2}\left(-v_1^{*\prime}\right)\right] \\
&\geq \frac{p^*-b_1}{\left(v_1^*+a_1\right)^2}\left(-v_1^{*\prime}\right) > 0,
\end{aligned} \quad (123)$$

which proves Eq. (121). On account of that,

$$\frac{dp^*}{d\beta} = -\frac{\frac{\partial \mathscr{F}}{\partial \beta}}{\frac{\partial \mathscr{F}}{\partial p^*}} < 0, \quad \dots \quad (124)$$

where $\frac{\partial \mathscr{F}}{\partial \beta}$ and $\frac{\partial \mathscr{F}}{\partial p^*}$ are continuous with respect to β. Hence $p^*(\beta)$ is continuously differentiable and strictly decreasing with respect to β, $\beta \in (0, 1]$.

Now, since

$$G^*(\beta) = G(p^*(\beta)) = -Kp^*(\beta) + T, \quad \dots \quad (125)$$

and $p^*(\beta)$ is continuously differentiable and strictly decreasing with respect to β, and K and T are positive constants, then $G^*(\beta)$ is continuously differentiable and strictly increasing with respect to β, $\beta \in (0, 1]$.

Now, we are going to show that $q_1^*(\beta)$ is continuously differentiable and strictly decreasing with respect to β. To do that, we first solve Eq. (114) for p^* to obtain the following equality:

$$\begin{aligned}
p^* &= \frac{\frac{b_0}{(1-\beta)v_0^*+a_0} + \frac{v_0^*+a_0}{(1-\beta)v_0^*+a_0}\left(\frac{b_1}{v_1^*+a_1}\right) + T + D}{\frac{1}{(1-\beta)v_0^*+a_0} + \frac{v_0^*+a_0}{(1-\beta)v_0^*+a_0}\left(\frac{1}{v_1^*+a_1}\right) + K} \\
&= \frac{\left(v_0^*+a_0\right)b_1+\left(v_1^*+a_1\right)b_0+\left[(1-\beta)v_0^*+a_0\right]\left(v_1^*+a_1\right)(T+D)}{\left(v_0^*+a_0\right)+\left(v_1^*+a_1\right)+\left[(1-\beta)v_0^*+a_0\right]\left(v_1^*+a_1\right)K}.
\end{aligned} \quad (126)$$

We substitute Eq. (126) in $q_1^* = q_1(p^*; v_0^*, v_1^*)$, to deduce

$$\begin{aligned}
q_1^* &= \frac{p^*-b_1}{v_1^*+a_1} \\
&= \frac{\frac{\left(v_0^*+a_0\right)b_1+\left(v_1^*+a_1\right)b_0+\left[(1-\beta)v_0^*+a_0\right]\left(v_1^*+a_1\right)(T+D)}{\left(v_0^*+a_0\right)+\left(v_1^*+a_1\right)+\left[(1-\beta)v_0^*+a_0\right]\left(v_1^*+a_1\right)K} - b_1}{v_1^*+a_1} \\
&= \frac{-(b_1-b_0)+\left[(1-\beta)v_0^*+a_0\right](G(b_1)+D)}{\left(v_0^*+a_0\right)+\left(v_1^*+a_1\right)\left\{1+\left[(1-\beta)v_0^*+a_0\right]K\right\}}.
\end{aligned} \quad (127)$$

By plugging Eq. (72) in Eq. (127) we have that

$$\begin{aligned}
q_1^* &= \frac{-(b_1-b_0)+\left[(1-\beta)v_0^*+a_0\right](G(b_1)+D)}{\left(v_0^*+a_0\right)+\left[\frac{(1-\beta)v_0^*+a_0}{1+\left[(1-\beta)v_0^*+a_0\right]K}+a_1\right]\left\{1+\left[(1-\beta)v_0^*+a_0\right]K\right\}} \\
&= \frac{-(b_1-b_0)+\left[(1-\beta)v_0^*+a_0\right](G(b_1)+D)}{\left(v_0^*+a_0+a_1\right)+\left[(1-\beta)v_0^*+a_0\right](1+a_1K)} = \frac{M}{N},
\end{aligned} \quad (128)$$

where

$$M = M(\beta) = -(b_1-b_0)+\left[(1-\beta)v_0^*+a_0\right](G(b_1)+D) \quad \dots \quad (129)$$

and

$$N = N(\beta) = \left(v_0^*+a_0+a_1\right)+\left[(1-\beta)v_0^*+a_0\right](1+a_1K). \quad \dots \quad (130)$$

It is easy to see that M and N are continuously differentiable with respect to β with

$$M' = \left[-v_0^*+(1-\beta)v_0^{*\prime}\right](G(b_1)+D), \quad \dots \quad (131)$$

$$N' = v_0^{*\prime} + \left[-v_0^*+(1-\beta)v_0^{*\prime}\right](1+a_1K). \quad \dots \quad (132)$$

Moreover, $N > 0$, so q_1^* is continuously differentiable with respect to β and

$$q_1^{*\prime} = \frac{M'N - MN'}{N^2}. \quad \dots \quad (133)$$

Thus, to find the value of $q_1^{*\prime}$ it suffices to estimate the value of the numerator of Eq. (133):

$$\begin{aligned}
&M'N - MN' \\
&= \left[-v_0^*+(1-\beta)v_0^{*\prime}\right](G(b_1)+D)\left\{\left(v_0^*+a_0+a_1\right)+\right. \\
&\quad \left.+\left[(1-\beta)v_0^*+a_0\right](1+a_1K)\right\} \\
&\quad -\left\{-(b_1-b_0)+\left[(1-\beta)v_0^*+a_0\right](G(b_1)+D)\right\}\left\{v_0^{*\prime}+\right. \\
&\quad \left.+\left[-v_0^*+(1-\beta)v_0^{*\prime}\right](1+a_1K)\right\} \\
&= \left[\left(v_0^*+a_0\right)\left(-v_0^*-\beta v_0^{*\prime}\right)+\beta v_0^* v_0^{*\prime}\right](G(b_1)+D) \\
&\quad +a_1\left[-v_0^*+(1-\beta)v_0^{*\prime}\right](G(b_1)+D) \\
&\quad +\left\{v_0^{*\prime}+\left[-v_0^*+(1-\beta)v_0^{*\prime}\right](1+a_1K)\right\}(b_1-b_0).
\end{aligned} \quad (134)$$

Given the values of $a_0, a_1, b_0, b_1, \beta, v_0^*, v_0^{*\prime}, G(p), D$, and Eq. (98), it is transparent that Eq. (134) is non-positive. Moreover,

$$\begin{aligned}
M'N - MN' &= \left[\left(v_0^*+a_0\right)\left(-v_0^*-\beta v_0^{*\prime}\right)+\beta v_0^* v_0^{*\prime}\right](G(b_1)+D) \\
&\quad +a_1\left[-v_0^*+(1-\beta)v_0^{*\prime}\right](G(b_1)+D) \\
&\quad +\left\{v_0^{*\prime}+\left[-v_0^*+(1-\beta)v_0^{*\prime}\right](1+a_1K)\right\}(b_1-b_0) \\
&\leq a_1\left[-v_0^*+(1-\beta)v_0^{*\prime}\right](G(b_1)+D) < 0.
\end{aligned} \quad (135)$$

Thus,

$$M'N - MN' < 0, \quad \dots \quad (136)$$

which proves that $q_1^{*\prime} < 0$, so $q_1^*(\beta)$ is continuously differentiable and strictly decreasing with respect to β, $\beta \in$

$(0,1]$.

Finally, since

$$q_0^*(\beta)+q_1^*(\beta)=G^*(\beta)+D, \quad (137)$$

then

$$q_0^*(\beta)=-q_1^*(\beta)+G^*(\beta)+D. \quad (138)$$

And since $G^*(\beta)$ is continuously differentiable and strictly increasing with respect to β, the function $q_1^*(\beta)$ is continuously differentiable and strictly decreasing with respect to β. Because D is constant, we have that $q_0^*(\beta)$ is continuously differentiable and strictly increasing with respect to β, $\beta\in(0,1]$. The proof of the theorem is complete.

B.2. Proof of Theorem 5

Theorem 5. *For the linear (affine) demand function $G(p)$ described in Eq. (17), the price $p^c(\beta)$ and the supply values $q_i^c(\beta)$, $i=0,1$, from the Cournot-Nash equilibrium, are continuously differentiable with respect to $\beta\in(0,1]$. Furthermore, $p^c(\beta)$ and $q_1^c(\beta)$ strictly decrease, whereas $q_0^c(\beta)$ strictly grows along with β.*

Proof. Let's consider the exterior equilibrium (p^c,q_0^c,q_1^c), i.e., such a vector that the following equalities hold:

$$q_0^c+q_1^c=G(p^c)+D, \quad (139)$$

$$q_0^c=\frac{p^c-b_0}{(1-\beta)\frac{1}{K}+a_0}+\frac{\beta\frac{1}{K}}{(1-\beta)\frac{1}{K}+a_0}\left(\frac{p^c-b_1}{\frac{1}{K}+a_1}\right), \quad (140)$$

and

$$q_1^c=\frac{p^c-b_1}{\frac{1}{K}+a_1}, \quad (141)$$

where

$$G(p^c)=-Kp^c+T. \quad (142)$$

From Eq. (139) one has

$$q_0^c+q_1^c-G(p^c)-D=0. \quad (143)$$

By substituting Eqs. (140), (141), and (142) in Eq. (143), similarly to (32), we have

$$\begin{aligned}q_0^c+q_1^c-G(p^c)-D&=\frac{p^c-b_0}{(1-\beta)\frac{1}{K}+a_0}+\frac{\beta\frac{1}{K}}{(1-\beta)\frac{1}{K}+a_0}\left(\frac{p^c-b_1}{\frac{1}{K}+a_1}\right)\\&\quad+\frac{p^c-b_1}{\frac{1}{K}+a_1}+Kp^c-T-D\\&=p^c\left[\frac{1}{(1-\beta)\frac{1}{K}+a_0}+\frac{\frac{1}{K}+a_0}{(1-\beta)\frac{1}{K}+a_0}\left(\frac{1}{\frac{1}{K}+a_1}\right)\right]\\&\quad-\left[\frac{b_0}{(1-\beta)\frac{1}{K}+a_0}+\frac{\frac{1}{K}+a_0}{(1-\beta)\frac{1}{K}+a_0}\left(\frac{b_1}{\frac{1}{K}+a_1}\right)\right]\\&\quad+Kp^c-T-D=0.\end{aligned} \quad (144)$$

Solving Eq. (144) for p^c, similarly to Eq. (126), we get the equation

$$\begin{aligned}p^c&=\frac{\frac{b_0}{(1-\beta)\frac{1}{K}+a_0}+\frac{\frac{1}{K}+a_0}{(1-\beta)\frac{1}{K}+a_0}\left(\frac{b_1}{\frac{1}{K}+a_1}\right)+T+D}{\frac{1}{(1-\beta)\frac{1}{K}+a_0}+\frac{\frac{1}{K}+a_0}{(1-\beta)\frac{1}{K}+a_0}\left(\frac{1}{\frac{1}{K}+a_1}\right)+K}\\&=\frac{\left(\frac{1}{K}+a_0\right)b_1+\left(\frac{1}{K}+a_1\right)b_0+\left[(1-\beta)\frac{1}{K}+a_0\right]\left(\frac{1}{K}+a_1\right)(T+D)}{\left(\frac{1}{K}+a_0\right)+\left(\frac{1}{K}+a_1\right)+\left[(1-\beta)\frac{1}{K}+a_0\right]\left(\frac{1}{K}+a_1\right)K}=\frac{X}{Y},\end{aligned} \quad (145)$$

where

$$\begin{aligned}X(\beta)&=\left(\frac{1}{K}+a_0\right)b_1+\left(\frac{1}{K}+a_1\right)b_0\\&\quad+\left[(1-\beta)\frac{1}{K}+a_0\right]\left(\frac{1}{K}+a_1\right)(T+D),\end{aligned} \quad (146)$$

and

$$\begin{aligned}Y(\beta)&=\left(\frac{1}{K}+a_0\right)+\left(\frac{1}{K}+a_1\right)\\&\quad+\left[(1-\beta)\frac{1}{K}+a_0\right]\left(\frac{1}{K}+a_1\right)K.\end{aligned} \quad (147)$$

It's easy to see that X and Y are continuously differentiable with respect to β with

$$X'=-\frac{1}{K}\left(\frac{1}{K}+a_1\right)(T+D), \quad (148)$$

$$Y'=-\left(\frac{1}{K}+a_1\right). \quad (149)$$

Moreover, $Y>0$, whence p^c is continuously differentiable with respect to β with

$$p^{c\prime}=\frac{X'Y-XY'}{Y^2}. \quad (150)$$

To compute the value of $p^{c\prime}$ it is sufficient to calculate the value of the numerator of Eq. (150):

$$\begin{aligned}&X'Y-XY'\\&=-\tfrac{1}{K}\left(\tfrac{1}{K}+a_1\right)(T+D)\left\{\left(\tfrac{1}{K}+a_0\right)+\left(\tfrac{1}{K}+a_1\right)\right.\\&\quad\left.+\left[(1-\beta)\tfrac{1}{K}+a_0\right]\left(\tfrac{1}{K}+a_1\right)K\right\}\\&\quad-\left[-\left(\tfrac{1}{K}+a_1\right)\right]\left\{\left(\tfrac{1}{K}+a_0\right)b_1+\left(\tfrac{1}{K}+a_1\right)b_0\right.\\&\quad\left.+\left[(1-\beta)\tfrac{1}{K}+a_0\right]\left(\tfrac{1}{K}+a_1\right)(T+D)\right\}\\&=-\tfrac{1}{K}\left(\tfrac{1}{K}+a_0\right)\left(\tfrac{1}{K}+a_1\right)(G(b_1)+D)\\&\quad-\tfrac{1}{K}\left(\tfrac{1}{K}+a_1\right)^2(G(b_0)+D).\end{aligned} \quad (151)$$

Given the values of $a_0,a_1,K,G(p)$, and D, it is clear that Eq. (151) is non-positive. Moreover,

$$\begin{aligned}X'Y-XY'&=-\tfrac{1}{K}\left(\tfrac{1}{K}+a_0\right)\left(\tfrac{1}{K}+a_1\right)(G(b_1)+D)\\&\quad-\tfrac{1}{K}\left(\tfrac{1}{K}+a_1\right)^2(G(b_0)+D)\\&\le-\tfrac{1}{K}\left(\tfrac{1}{K}+a_0\right)\left(\tfrac{1}{K}+a_1\right)(G(b_1)+D)<0.\end{aligned} \quad (152)$$

Then,

$$X'Y-XY'<0, \quad (153)$$

which proves that $p^{c\prime}<0$, so $p^c(\beta)$ is continuously differentiable and strictly decreasing with respect to β, $\beta\in$

$(0,1]$.

Since

$$q_1^c(\beta) = \frac{p^c(\beta) - b_1}{\frac{1}{K} + a_1}, \quad (154)$$

and $p^c(\beta)$ is continuously differentiable and strictly decreasing with respect to β, and a_1, b_1 and K are positive constants, then $q_1^c(\beta)$ is continuously differentiable and strictly decreasing with respect to $\beta \in (0,1]$.

Finally, since

$$q_0^c(\beta)+q_1^c(\beta)=G(p^c(\beta))+D=-Kp^c(\beta)+T+D, \quad (155)$$

then,

$$q_0^c(\beta) = -q_1^c(\beta) - Kp^c(\beta) + T + D. \quad (156)$$

And as $q_1^c(\beta)$ is continuously differentiable and strictly decreasing with respect to β, the function $p^c(\beta)$ also-has the same property, and K, T and D are non-negative constants, then $q_0^*(\beta)$ is continuously differentiable and strictly increasing with respect to $\beta \in (0,1]$. The proof of the theorem is complete.

B.3. Proof of Theorem 6

Theorem 6. *If the demand function $G(p)$ is depicted as in Eq.* (17)*, the price $p^t(\beta)$ and the output volumes $q_i^t(\beta)$, $i = 0,1$, related to the perfect competition equilibrium, are invariant for all $\beta \in (0,1]$ and are described by the clear-cut expressions:*

$$p^t = \frac{a_0 b_1 + a_1 b_0 + a_0 a_1 (T+D)}{a_0 + a_1 + a_0 a_1 K}; \quad (157)$$

$$q_0^t = \frac{a_1 (G(b_0) + D) + (b_1 - b_0)}{a_0 + a_1 + a_0 a_1 K}; \quad (158)$$

$$q_1^t = \frac{a_0 (G(b_1) + D) - (b_1 - b_0)}{a_0 + a_1 + a_0 a_1 K}. \quad (159)$$

Proof. Let us consider the exterior equilibrium (p^t, q_0^t, q_1^t), i.e., such a vector that the following equalities hold:

$$q_0^t + q_1^t = G(p^t) + D, \quad (160)$$

$$q_0^t = \frac{p^t - b_0}{a_0}, \quad (161)$$

$$q_1^t = \frac{p^t - b_1}{a_1}, \quad (162)$$

where

$$G(p^t) = -Kp^t + T. \quad (163)$$

From Eq. (139) one gets that

$$q_0^t + q_1^t - G(p^t) - D = 0. \quad (164)$$

Next, by plugging Eqs. (161), (162), and (163) in Eq. (164), we deduce that

$$\begin{aligned} q_0^t+q_1^t-G(p^t)-D &= \frac{p^t-b_0}{a_0} + \frac{p^t-b_1}{a_1} + Kp^t - T - D \\ &= p^t\left(\frac{1}{a_0}+\frac{1}{a_1}\right) - \left(\frac{b_1}{a_1}+\frac{b_0}{a_0}\right) + Kp^c - T - D = 0. \end{aligned} \quad (165)$$

By solving Eq. (165) for p^t, we obtain the equality

$$\begin{aligned} p^t &= \frac{\frac{b_1}{a_1} + \frac{b_0}{a_0} + T + D}{\frac{1}{a_0} + \frac{1}{a_1} + K} \\ &= \frac{a_0 b_1 + a_1 b_0 + a_0 a_1 (T+D)}{a_0 + a_1 + a_0 a_1 K}, \end{aligned} \quad (166)$$

showing that the function $p^t(\beta)$ is constant for all $\beta \in (0,1]$.

Moreover, since

$$\begin{aligned} q_0^t &= \frac{p^t - b_0}{a_0} = \frac{\frac{a_0 b_1 + a_1 b_0 + a_0 a_1 (T+D)}{a_0 + a_1 + a_0 a_1 K} - b_0}{a_0} \\ &= \frac{a_1 (G(b_0) + D) + (b_1 - b_0)}{a_0 + a_1 + a_0 a_1 K}, \end{aligned} \quad (167)$$

and

$$\begin{aligned} q_1^t &= \frac{p^t - b_1}{a_1} = \frac{\frac{a_0 b_1 + a_1 b_0 + a_0 a_1 (T+D)}{a_0 + a_1 + a_0 a_1 K} - b_1}{a_1} \\ &= \frac{a_0 (G(b_1) + D) - (b_1 - b_0)}{a_0 + a_1 + a_0 a_1 K}, \end{aligned} \quad (168)$$

the functions $q_0^t(\beta)$ and $q_1^t(\beta)$ are constant for all $\beta \in (0,1]$. The proof of the theorem is complete.

Appendix C.

C.1. Proof of Theorem 7

Theorem 7. *For the linear (affine) function $G(p)$ described in Eq.* (17)*, the price functions in the consistent CVE, $p^*(\beta)$, the Cournot-Nash equilibrium, $p^c(\beta)$, and the perfect competition equilibrium, p^t, satisfy the following inequalities:*

$$p^t < \lim_{\beta \to 0} p^*(\beta) \quad (169)$$

and

$$p^*(\beta) < p^c(\beta) \text{ for all } \beta \in (0,1]. \quad (170)$$

Proof. First, we prove inequality (21):

$$p^t < \lim_{\beta \to 0} p^*(\beta).$$

Introduce the following notation:

$$\begin{aligned} \widehat{v_0^*} &= \lim_{\beta \to 0} v_0^*(\beta) \\ &= \frac{2(a_0+a_1+a_0a_1K)}{\left(2a_0K+a_0a_1K^2\right)+\sqrt{\left(2a_0K+a_0a_1K^2\right)^2+4\left(2K+a_1K^2\right)(a_0+a_1+a_0a_1K)}} \end{aligned} \quad (171)$$

$$\begin{aligned} \widehat{v_1^*} &= \lim_{\beta \to 0} v_1^*(\beta) = \lim_{\beta \to 0} \frac{(1-\beta)v_0^*+a_0}{1+\left[(1-\beta)v_0^*+a_0\right]K} \\ &= \frac{\widehat{v_0^*}+a_0}{1+\left(\widehat{v_0^*}+a_0\right)K} > 0. \end{aligned} \quad (172)$$

Therefore,

$$\lim_{\beta\to 0} p^*(\beta) = \lim_{\beta\to 0} \frac{(v_0^*+a_0)b_1+(v_1^*+a_1)b_0+[(1-\beta)v_0^*+a_0](v_1^*+a_1)(T+D)}{(v_0^*+a_0)+(v_1^*+a_1)+[(1-\beta)v_0^*+a_0](v_1^*+a_1)K} = \frac{(\widehat{v_0^*}+a_0)b_1+(\widehat{v_1^*}+a_1)b_0+(\widehat{v_0^*}+a_0)(\widehat{v_1^*}+a_1)(T+D)}{(\widehat{v_0^*}+a_0)+(\widehat{v_1^*}+a_1)+(\widehat{v_0^*}+a_0)(\widehat{v_1^*}+a_1)K}. \quad (173)$$

Now, we compute the difference

$$\begin{aligned}&\lim_{\beta\to 0} p^*(\beta)-p^t\\ &= \frac{(\widehat{v_0^*}+a_0)b_1+(\widehat{v_1^*}+a_1)b_0+(\widehat{v_0^*}+a_0)(\widehat{v_1^*}+a_1)(T+D)}{(\widehat{v_0^*}+a_0)+(\widehat{v_1^*}+a_1)+(\widehat{v_0^*}+a_0)(\widehat{v_1^*}+a_1)K} - \frac{a_0b_1+a_1b_0+a_0a_1(T+D)}{a_0+a_1+a_0a_1K}\\ &= \frac{[(\widehat{v_0^*}+a_0)b_1+(\widehat{v_1^*}+a_1)b_0+(\widehat{v_0^*}+a_0)(\widehat{v_1^*}+a_1)(T+D)](a_0+a_1+a_0a_1K)}{[(\widehat{v_0^*}+a_0)+(\widehat{v_1^*}+a_1)+(\widehat{v_0^*}+a_0)(\widehat{v_1^*}+a_1)K](a_0+a_1+a_0a_1K)}\\ &\quad - \frac{[(\widehat{v_0^*}+a_0)+(\widehat{v_1^*}+a_1)+(\widehat{v_0^*}+a_0)(\widehat{v_1^*}+a_1)K][a_0b_1+a_1b_0+a_0a_1(T+D)]}{[(\widehat{v_0^*}+a_0)+(\widehat{v_1^*}+a_1)+(\widehat{v_0^*}+a_0)(\widehat{v_1^*}+a_1)K](a_0+a_1+a_0a_1K)}\\ &= \frac{R1}{R2},\end{aligned} \quad (174)$$

where

$$\begin{aligned}R1=&\Big[(\widehat{v_0^*}+a_0)b_1+(\widehat{v_1^*}+a_1)b_0\\ &+(\widehat{v_0^*}+a_0)(\widehat{v_1^*}+a_1)(T+D)\Big](a_0+a_1+a_0a_1K)\\ &-\Big[(\widehat{v_0^*}+a_0)+(\widehat{v_1^*}+a_1)\\ &+(\widehat{v_0^*}+a_0)(\widehat{v_1^*}+a_1)K\Big][a_0b_1+a_1b_0+a_0a_1(T+D)],\end{aligned} \quad (175)$$

and

$$\begin{aligned}R2=&\Big[(\widehat{v_0^*}+a_0)+(\widehat{v_1^*}+a_1)\\ &+(\widehat{v_0^*}+a_0)(\widehat{v_1^*}+a_1)K\Big](a_0+a_1+a_0a_1K).\end{aligned} \quad . . . . (176)$$

Given the values of $a_0, a_1, \widehat{v_0^*}, \widehat{v_1^*}$, and K, it is easy to see that $R2 > 0$. Hence, to calculate the value of Eq. (174), it is enough to estimate the value of Eq. (175). That is,

$$\begin{aligned}R1=&\Big[(\widehat{v_0^*}+a_0)b_1+(\widehat{v_1^*}+a_1)b_0\\ &+(\widehat{v_0^*}+a_0)(\widehat{v_1^*}+a_1)(T+D)\Big](a_0+a_1+a_0a_1K)\\ &-\Big[(\widehat{v_0^*}+a_0)+(\widehat{v_1^*}+a_1)\\ &+(\widehat{v_0^*}+a_0)(\widehat{v_1^*}+a_1)K\Big][a_0b_1+a_1b_0+a_0a_1(T+D)]\\ =&a_0\widehat{v_1^*}\Big[(\widehat{v_0^*}+a_0)(G(b_1)+D)-(b_1-b_0)\Big]\\ &+a_1\widehat{v_0^*}\Big[(\widehat{v_1^*}+a_1)(G(b_0)+D)+(b_1-b_0)\Big].\end{aligned} \quad . \quad (177)$$

Given the values of $a_0, a_1, b_0, b_1, \widehat{v_0^*}, \widehat{v_1^*}, G(p), D$, and assuming **A3**, it is trivial that Eq. (177) is non-negative. Moreover,

$$\begin{aligned}R1=&a_0\widehat{v_1^*}\Big[(\widehat{v_0^*}+a_0)(G(b_1)+D)-(b_1-b_0)\Big]\\ &+a_1\widehat{v_0^*}\Big[(\widehat{v_1^*}+a_1)(G(b_0)+D)+(b_1-b_0)\Big]\\ \ge&a_1\widehat{v_0^*}\Big[(\widehat{v_1^*}+a_1)(G(b_0)+D)+(b_1-b_0)\Big]\\ \ge&a_1\widehat{v_0^*}(\widehat{v_1^*}+a_1)(G(b_0)+D)\\ \ge&a_1^2\widehat{v_0^*}(G(b_0)+D)\ge a_1^2\widehat{v_0^*}G(b_0)>0.\end{aligned} \quad . . . . (178)$$

And since $R1 > 0$, by Eq. (178), then

$$\lim_{\beta\to 0} p^*(\beta) - p^t > 0, \quad . . . . (179)$$

which proves inequality (21).

Now, we establish inequality (22):

$p^*(\beta) < p^c(\beta)$ para todo $\beta \in (0,1]$.

In order to do that, we introduce the following notation:

$$v_i^* = v_i^*(\beta),\ i = 0,1.$$

From Eqs. (72) and (73) it's easy to see that the following inequality hold for all $\beta \in (0,1]$:

$$v_i^* < \frac{1}{K},\ i = 0,1. \quad . . . . (180)$$

Now, we compute the difference

$$\begin{aligned}&(p^c-p^*)(\beta)\\ &= \frac{(\frac{1}{K}+a_0)b_1+(\frac{1}{K}+a_1)b_0+[(1-\beta)\frac{1}{K}+a_0](\frac{1}{K}+a_1)(T+D)}{(\frac{1}{K}+a_0)+(\frac{1}{K}+a_1)+[(1-\beta)\frac{1}{K}+a_0](\frac{1}{K}+a_1)K}\\ &\quad - \frac{(v_0^*+a_0)b_1+(v_1^*+a_1)b_0+[(1-\beta)v_0^*+a_0](v_1^*+a_1)(T+D)}{(v_0^*+a_0)+(v_1^*+a_1)+[(1-\beta)v_0^*+a_0](v_1^*+a_1)K}\\ &= \frac{S1}{S2},\end{aligned} \quad (181)$$

where

$$\begin{aligned}S1 =& \{(\tfrac{1}{K}+a_0)b_1+(\tfrac{1}{K}+a_1)b_0+[(1-\beta)\tfrac{1}{K}+a_0](\tfrac{1}{K}+a_1)(T+D)\}\cdot\\ &\cdot\{(v_0^*+a_0)+(v_1^*+a_1)+[(1-\beta)v_0^*+a_0](v_1^*+a_1)K\}\\ &-\{(\tfrac{1}{K}+a_0)+(\tfrac{1}{K}+a_1)+[(1-\beta)\tfrac{1}{K}+a_0](\tfrac{1}{K}+a_1)K\}\cdot\\ &\cdot\{(v_0^*+a_0)b_1+(v_1^*+a_1)b_0+[(1-\beta)v_0^*+a_0](v_1^*+a_1)(T+D)\},\end{aligned}$$
$$. . . . . (182)$$

and

$$\begin{aligned}S2=&\{(\tfrac{1}{K}+a_0)+(\tfrac{1}{K}+a_1)+[(1-\beta)\tfrac{1}{K}+a_0](\tfrac{1}{K}+a_1)K\}\cdot\\ &\cdot\{(v_0^*+a_0)+(v_1^*+a_1)+[(1-\beta)v_0^*+a_0](v_1^*+a_1)K\}.\end{aligned} \quad (183)$$

For any fixed values of $a_0, a_1, \beta, v_0^*, v_1^*$, and K, it is apparent that $S2 > 0$. Because of that, in order to find the value of Eq. (181), it suffices to calculate the value of Eq. (182). So,

$$\begin{aligned}S1=&\{[(1-\beta)\tfrac{1}{K}+a_0](v_0^*+a_0)(\tfrac{1}{K}+a_1)\\ &-[(1-\beta)v_0^*+a_0](v_1^*+a_1)(\tfrac{1}{K}+a_0)\}(G(b_1)+D)\\ &+\{[(1-\beta)\tfrac{1}{K}+a_0](v_1^*+a_1)(\tfrac{1}{K}+a_1)\\ &-[(1-\beta)v_0^*+a_0](v_1^*+a_1)(\tfrac{1}{K}+a_1)\}(G(b_0)+D)\\ &+[(v_1^*+a_1)(\tfrac{1}{K}+a_0)-(v_0^*+a_0)(\tfrac{1}{K}+a_1)](b_1-b_0)\\ =&X_1(G(b_1)+D)+X_2(G(b_0)+D)+X_3(b_1-b_0),\end{aligned} \quad . \quad (184)$$

where

$$\begin{aligned}X_1 =& \left[(1-\beta)\frac{1}{K}+a_0\right](v_0^*+a_0)\left(\frac{1}{K}+a_1\right)\\ &-[(1-\beta)v_0^*+a_0](v_1^*+a_1)\left(\frac{1}{K}+a_0\right),\end{aligned} \quad (185)$$

$$X_2 = \left[(1-\beta)\frac{1}{K}+a_0\right](v_1^*+a_1)\left(\frac{1}{K}+a_1\right) - [(1-\beta)v_0^*+a_0](v_1^*+a_1)\left(\frac{1}{K}+a_1\right) \tag{186}$$

and

$$X_3 = (v_1^*+a_1)\left(\frac{1}{K}+a_0\right) - (v_0^*+a_0)\left(\frac{1}{K}+a_1\right). \quad \ldots\ldots\ldots \tag{187}$$

Now, for any given values of $a_0, a_1, \beta, v_0^*, v_1^*, K$, and Eq. (180), one finds

$$\begin{aligned} X_2 &= \left[(1-\beta)\frac{1}{K}+a_0\right](v_1^*+a_1)\left(\frac{1}{K}+a_1\right) \\ &\quad - [(1-\beta)v_0^*+a_0](v_1^*+a_1)\left(\frac{1}{K}+a_1\right) \\ &= (1-\beta)\left(\frac{1}{K}-v_0^*\right)(v_1^*+a_1)\left(\frac{1}{K}+a_1\right) \geq 0 \end{aligned} \tag{188}$$

for all $\beta \in (0,1]$.

Now, we are going to show that $X_1 > 0$ for all $\beta \in (0,1]$. Plugging Eq. (72) in X_1, we obtain

$$\begin{aligned} X_1 &= \left[(1-\beta)\tfrac{1}{K}+a_0\right]\left(v_0^*+a_0\right)\left(\tfrac{1}{K}+a_1\right) \\ &\quad - \left[(1-\beta)v_0^*+a_0\right]\left(\frac{(1-\beta)v_0^*+a_0}{1+\left[(1-\beta)v_0^*+a_0\right]K}+a_1\right)\left(\tfrac{1}{K}+a_0\right) \\ &= \frac{\left[(1-\beta)\frac{1}{K}+a_0\right]\left(v_0^*+a_0\right)\left(\frac{1}{K}+a_1\right)\left\{1+\left[(1-\beta)v_0^*+a_0\right]K\right\}}{1+\left[(1-\beta)v_0^*+a_0\right]K} \\ &\quad - \frac{\left[(1-\beta)v_0^*+a_0\right]\left[(1-\beta)v_0^*+a_0+a_1\left\{1+\left[(1-\beta)v_0^*+a_0\right]K\right\}\right]\left(\frac{1}{K}+a_0\right)}{1+\left[(1-\beta)v_0^*+a_0\right]K} \\ &= \tfrac{T1}{T2}, \end{aligned} \tag{189}$$

where

$$\begin{aligned} T1 &= \left[(1-\beta)\tfrac{1}{K}+a_0\right]\left(v_0^*+a_0\right)\left(\tfrac{1}{K}+a_1\right)\left\{1+\left[(1-\beta)v_0^*+a_0\right]K\right\} \\ &\quad - \left[(1-\beta)v_0^*+a_0\right]\left[(1-\beta)v_0^*+a_0 \right. \\ &\quad \left. + a_1\left\{1+\left[(1-\beta)v_0^*+a_0\right]K\right\}\right]\left(\tfrac{1}{K}+a_0\right), \end{aligned} \tag{190}$$

and

$$T2 = 1+[(1-\beta)v_0^*+a_0]K. \quad \ldots\ldots\ldots \tag{191}$$

For any fixed values of a_0, β, v_0^*, and K, it is transparent that $T2 > 0$. Therefore, to compute the value of Eq. (189), we need to calculate the value of $T1$:

$$\begin{aligned} T1 &= (1-\beta)^2\left(\tfrac{1}{K}+a_1\right)\left(\tfrac{1}{K}-v_0^*\right)a_0Kv_0^*+(1-\beta)\left[\left(\tfrac{1}{K}+a_1\right)\left(v_0^*+a_0\right)\tfrac{1}{K}\right. \\ &\quad + \left(\tfrac{1}{K}+a_1\right)\left(v_0^*+a_0\right)\left(\tfrac{1}{K}+v_0^*\right)a_0K \\ &\quad \left. - \left(\tfrac{1}{K}+a_0\right)a_1v_0^*-2\left(\tfrac{1}{K}+a_0\right)(1+a_1K)a_0v_0^*\right] \\ &\quad + \left(\tfrac{1}{K}+a_0\right)a_0\left[\left(\tfrac{1}{K}+a_1\right)Kv_0^*-a_1\right] \\ &= (1-\beta)^2Y_1+(1-\beta)Y_2+\left(\tfrac{1}{K}+a_0\right)a_0Y_3, \end{aligned} \tag{192}$$

where

$$Y_1 = \left(\frac{1}{K}+a_1\right)\left(\frac{1}{K}-v_0^*\right)a_0Kv_0^*, \quad \ldots \tag{193}$$

$$\begin{aligned} Y_2 &= \left(\tfrac{1}{K}+a_1\right)\left(v_0^*+a_0\right)\tfrac{1}{K}+\left(\tfrac{1}{K}+a_1\right)\left(v_0^*+a_0\right)\left(\tfrac{1}{K}+v_0^*\right)a_0K \\ &\quad - \left(\tfrac{1}{K}+a_0\right)a_1v_0^*-2\left(\tfrac{1}{K}+a_0\right)(1+a_1K)a_0v_0^*, \end{aligned} \tag{194}$$

and

$$Y_3 = \left(\frac{1}{K}+a_1\right)Kv_0^*-a_1. \quad \ldots\ldots\ldots \tag{195}$$

For any fixed values of a_0, a_1, v_0^*, K, and Eq. (180), one concludes that $Y_1 > 0$, and $Y_3 = Y_3(\beta)$ strictly decreases by β, since $v_0^* = v_0^*(\beta)$ is strictly decreasing with respect to β, and $\left(\frac{1}{K}+a_1\right)K > 0$. Thus,

$$\begin{aligned} Y_3 &= Y_3(\beta) \geq Y_3(1) = \left(\frac{1}{K}+a_1\right)Kv_0^*(1)-a_1 \\ &= \frac{\left[(1+a_1K)^2-(2+a_1K)a_1K\right]a_0}{1+2a_0K+a_1K+a_0a_1K^2} \\ &= \frac{a_0}{1+2a_0K+a_1K+a_0a_1K^2} > 0. \end{aligned} \tag{196}$$

Then, $Y_3 > 0$ for all $\beta \in (0,1]$.

Now, we are going to show that $Y_2 > 0$ for all $\beta \in (0,1]$:

$$\begin{aligned} Y_2 &= \left(\tfrac{1}{K}+a_1\right)\left(v_0^*+a_0\right)\tfrac{1}{K}+\left(\tfrac{1}{K}+a_1\right)\left(v_0^*+a_0\right)\left(\tfrac{1}{K}+v_0^*\right)a_0K \\ &\quad - \left(\tfrac{1}{K}+a_0\right)a_1v_0^*-2\left(\tfrac{1}{K}+a_0\right)(1+a_1K)a_0v_0^* \\ &= \left[\left(\tfrac{1}{K}+a_1\right)a_0Kv_0^*+\left(\tfrac{1}{K^2}-a_0a_1\right)\right]v_0^* \\ &\quad + \left(\tfrac{1}{K}+a_0\right)\left(\tfrac{1}{K}+a_1\right)\left(\tfrac{1}{K}-v_0^*\right)a_0K \\ &= Z_1v_0^*+Z_2, \end{aligned} \tag{197}$$

where

$$Z_1 = \left(\frac{1}{K}+a_1\right)a_0Kv_0^*+\left(\frac{1}{K^2}-a_0a_1\right) \quad . \tag{198}$$

and

$$Z_2 = \left(\frac{1}{K}+a_0\right)\left(\frac{1}{K}+a_1\right)\left(\frac{1}{K}-v_0^*\right)a_0K. \tag{199}$$

Given the values of a_0, a_1, v_0^*, K, and Eq. (180), one has that $Z_2 > 0$ for all $\beta \in (0,1]$, and $Z_1 = Z_1(\beta)$ is strictly decreasing with respect to β, because $v_0^*(\beta)$ strictly decreases by β ,and $(a_1+\frac{1}{K})a_0K > 0$. Thus,

$$\begin{aligned} Z_1 &= Z_1(\beta) \geq Z_1(1) = \left(\tfrac{1}{K}+a_1\right)a_0Kv_0^*(1)+\left(\tfrac{1}{K^2}-a_0a_1\right) \\ &= \left(\tfrac{1}{K}+a_1\right)a_0K\frac{a_0+a_1+a_0a_1K}{1+2a_0K+a_1K+a_0a_1K^2}+\left(\tfrac{1}{K^2}-a_0a_1\right) \quad . \\ &= \frac{a_0^2}{1+2a_0K+a_1K+a_0a_1K^2}+\frac{1}{K^2} > 0. \end{aligned} \tag{200}$$

Then $Z_1 > 0$ for all $\beta \in (0,1]$, which proves that $Y_2 = v_0^*Z_1+Z_2 > 0$ for all $\beta \in (0,1]$.

Now, since $Y_1, Y_2, Y_3 > 0$, we have that

$$T1 = (1-\beta)^2Y_1+(1-\beta)Y_2+\left(\frac{1}{K}+a_0\right)a_0Y_3 > 0 \quad \ldots\ldots\ldots \tag{201}$$

which proves that

$$X_1 = \frac{T1}{T2} > 0. \quad \ldots\ldots\ldots \tag{202}$$

Since $X_1 > 0$ and $X_2 \geq 0$, then, if $X_3 \geq 0$ for $\beta_0 \in (0,1]$, we have that

$$S1{=}X_1(G(b_1){+}D){+}X_2(G(b_0){+}D){+}X_3(b_1{-}b_0){>}0, \quad (203)$$

for $\beta_0 \in (0,1]$.

On the other hand, if $X_3 < 0$ for $\beta_0 \in (0,1]$, then,

$$\begin{aligned}S1&{=}X_1(G(b_1){+}D){+}X_2(G(b_0){+}D){+}X_3(b_1{-}b_0)\\&{=}\left[(1{-}\beta)\tfrac{1}{K}{+}a_0\right]\left(v_0^*{+}a_0\right)\left(\tfrac{1}{K}{+}a_1\right)(G(b_1){+}D)\\&\quad{-}\left[(1{-}\beta)v_0^*{+}a_0\right]\left(v_1^*{+}a_1\right)\left(\tfrac{1}{K}{+}a_0\right)(G(b_1){+}D)\\&\quad{+}X_2(G(b_0){+}D){+}X_3(b_1{-}b_0)\\&{=}(1{-}\beta)\left[\tfrac{1}{K}\left(v_0^*{+}a_0\right)\left(\tfrac{1}{K}{+}a_1\right)\right.\\&\quad\left.{-}v_0^*\left(v_1^*{+}a_1\right)\left(\tfrac{1}{K}{+}a_0\right)\right](G(b_1){+}D)\\&\quad{-}a_0\left[\left(v_1^*{+}a_1\right)\left(\tfrac{1}{K}{+}a_0\right){-}\left(v_0^*{+}a_0\right)\left(\tfrac{1}{K}{+}a_1\right)\right](G(b_1){+}D)\\&\quad{+}X_2(G(b_0){+}D){+}X_3(b_1{-}b_0)\\&{=}(1{-}\beta)X_4(G(b_1){+}D){-}a_0X_3(G(b_1){+}D)\\&\quad{+}X_2(G(b_0){+}D){+}X_3(b_1{-}b_0)\\&{=}(1{-}\beta)X_4(G(b_1){+}D){-}X_3[a_0(G(b_1){+}D){-}(b_1{-}b_0)]\\&\quad{+}X_2(G(b_0){+}D),\end{aligned} \quad (204)$$

where

$$X_4{=}\tfrac{1}{K}\left(v_0^*{+}a_0\right)\left(\tfrac{1}{K}{+}a_1\right){-}v_0^*\left(v_1^*{+}a_1\right)\left(\tfrac{1}{K}{+}a_0\right). \quad (205)$$

Applying inequality Eq. (180) to Eq. (205), we see that

$$\begin{aligned}X_4&{=}\tfrac{1}{K}\left(v_0^*{+}a_0\right)\left(\tfrac{1}{K}{+}a_1\right){-}v_0^*\left(v_1^*{+}a_1\right)\left(\tfrac{1}{K}{+}a_0\right)\\&{>}\tfrac{1}{K}\left(v_0^*{+}a_0\right)\left(v_1^*{+}a_1\right){-}v_0^*\left(v_1^*{+}a_1\right)\left(\tfrac{1}{K}{+}a_0\right)\\&{=}a_0\left(v_1^*{+}a_1\right)\left(\tfrac{1}{K}{-}v_0^*\right){>}0.\end{aligned} \quad (206)$$

Thus, $X_4 > 0$ for $\beta_0 \in (0,1]$, and since $X_2 \geq 0$, $X_3 < 0$ and we assume **A3**, we have that

$$\begin{aligned}S1&{=}(1{-}\beta)X_4(G(b_1){+}D){-}X_3[a_0(G(b_1){+}D){-}(b_1{-}b_0)]\\&\quad{+}X_2(G(b_0){+}D)\\&{\geq}{-}X_3[a_0(G(b_1){+}D){-}(b_1{-}b_0)]{>}0,\end{aligned} \quad (207)$$

for $\beta_0 \in (0,1]$.

Therefore, $S1 > 0$ for all $\beta \in (0,1]$, that is,

$$(p^c - p^*)(\beta) = \frac{S1}{S2} > 0, \quad (208)$$

which finally proves Eq. (22). The proof of the theorem is complete.

C.2. Proof of Theorem 8

Theorem 8. *The functions $\pi_1^*(\beta)$ and $\pi_1^c(\beta)$ are strictly decreasing with respect to $\beta \in (0,1]$. Moreover, the following inequalities hold:*

$$\pi_1^*(1) > \pi_1^c(1) \quad (209)$$

and

$$\lim_{\beta\to 0} \pi_1^*(\beta) < \lim_{\beta\to 0} \pi_1^c(\beta). \quad (210)$$

Proof. First, we are going to show that π_1^* and π_1^c strictly decrease by β.

The function π_1^* is differentiable with respect to β and

$$\begin{aligned}\pi_1^{*\prime} &= \left(p^*q_1^* - \frac{1}{2}a_1{q_1^*}^2 - b_1q_1^*\right)'\\&= p^{*\prime}q_1^* + p^*q_1^{*\prime} - a_1q_1^*q_1^{*\prime} - b_1q_1^{*\prime}\\&= p^{*\prime}q_1^* + (p^* - a_1q_1^* - b_1)\,q_1^{*\prime}\\&= p^{*\prime}q_1^* + \left(p^* - b_1 - a_1\frac{p^*-b_1}{v_1^*+a_1}\right)q_1^{*\prime}\\&= p^{*\prime}q_1^* + \frac{v_1^*}{v_1^*+a_1}(p^*-b_1)\,q_1^{*\prime}.\end{aligned} \quad (211)$$

Given the values of $a_1, b_1, v_1^*, p^*, q_1^*, p^{*\prime}$ and $q_1^{*\prime}$, it's easy to see that

$$\pi_1^{*\prime} = p^{*\prime}q_1^* + \frac{v_1^*}{v_1^*+a_1}(p^*-b_1)\,q_1^{*\prime} < 0. \quad (212)$$

Similarly,

$$\pi_1^{c\prime} = p^{c\prime}q_1^c + \frac{\frac{1}{K}}{\frac{1}{K}+a_1}(p^c-b_1)\,q_1^{c\prime} < 0. \quad (213)$$

Because of that, π_1^* and π_1^c strictly decrease with respect to $\beta \in (0,1]$.

Now consider the difference of the functions π_1^* and π_1^c as follows:

$$\begin{aligned}\pi_1^c{-}\pi_1^*&{=}\left(p^cq_1^c{-}\tfrac{1}{2}a_1{q_1^c}^2{-}b_1q_1^c\right){-}\left(p^*q_1^*{-}\tfrac{1}{2}a_1{q_1^*}^2{-}b_1q_1^*\right)\\&{=}\left(p^c{-}b_1{-}\tfrac{1}{2}a_1q_1^c\right)q_1^c{-}\left(p^*{-}b_1{-}\tfrac{1}{2}a_1q_1^*\right)q_1^*\\&{=}\left[\left(\tfrac{1}{K}{+}a_1\right)\tfrac{p^c-b_1}{\frac{1}{K}+a_1}{-}\tfrac{1}{2}a_1q_1^c\right]q_1^c\\&\quad{-}\left[\left(v_1^*{+}a_1\right)\tfrac{p^*-b_1}{v_1^*+a_1}{-}\tfrac{1}{2}a_1q_1^*\right]q_1^*\\&{=}\left(\tfrac{1}{K}{+}\tfrac{1}{2}a_1\right){q_1^c}^2{-}\left(v_1^*{+}\tfrac{1}{2}a_1\right){q_1^*}^2.\end{aligned} \quad (214)$$

From Eq. (127) we have that

$$\begin{aligned}q_1^*&{=}\frac{-(b_1-b_0)+\left[(1-\beta)v_0^*+a_0\right](G(b_1)+D)}{\left(v_0^*+a_0\right)+\left(v_1^*+a_1\right)\left\{1+\left[(1-\beta)v_0^*+a_0\right]K\right\}}\\&{=}\frac{\left[(1-\beta)v_0^*+a_0\right](G(b_1)+D)-(b_1-b_0)}{\left(v_0^*+a_0\right)+\left(v_1^*+a_1\right)+\left[(1-\beta)v_0^*+a_0\right]\left(v_1^*+a_1\right)K},\end{aligned} \quad (215)$$

and similarly to Eqs. (127) and (215),

$$q_1^c{=}\frac{\left[(1-\beta)\frac{1}{K}+a_0\right](G(b_1)+D)-(b_1-b_0)}{\left(\frac{1}{K}+a_0\right)+\left(\frac{1}{K}+a_1\right)+\left[(1-\beta)\frac{1}{K}+a_0\right]\left(\frac{1}{K}+a_1\right)K}. \quad (216)$$

By substituting the expression of v_1^* given by Eq. (72) in Eq. (215) we have that

$$\begin{aligned}q_1^*&{=}\frac{\left[(1-\beta)v_0^*+a_0\right](G(b_1)+D)-(b_1-b_0)}{\left(v_0^*+a_0\right)+\left(v_1^*+a_1\right)+\left[(1-\beta)v_0^*+a_0\right]\left(v_1^*+a_1\right)K}\\&{=}\frac{\left[(1-\beta)v_0^*+a_0\right](G(b_1)+D)-(b_1-b_0)}{\left(v_0^*+a_0+a_1\right)+(1+a_1K)\left[(1-\beta)v_0^*+a_0\right]}.\end{aligned} \quad (217)$$

By Eq. (72),

$$v_1^* = \frac{(1-\beta)v_0^*+a_0}{1+\left[(1-\beta)v_0^*+a_0\right]K},$$

therefore,

$$\begin{aligned} v_1^* + \frac{1}{2}a_1 &= \frac{(1-\beta)v_0^* + a_0}{1+\left[(1-\beta)v_0^*+a_0\right]K} + \frac{1}{2}a_1 \\ &= \frac{\frac{1}{2}a_1 + \left(1+\frac{1}{2}a_1K\right)\left[(1-\beta)v_0^*+a_0\right]}{1+\left[(1-\beta)v_0^*+a_0\right]K}. \end{aligned} \tag{218}$$

On the other hand, from the expression for q_1^c obtained from Eq. (216) we have that

$$\begin{aligned} q_1^c &= \frac{\left[(1-\beta)\frac{1}{K}+a_0\right](G(b_1)+D)-(b_1-b_0)}{\left(\frac{1}{K}+a_0\right)+\left(\frac{1}{K}+a_1\right)+\left[(1-\beta)\frac{1}{K}+a_0\right]\left(\frac{1}{K}+a_1\right)K} \\ &= \frac{\left\{\left[(1-\beta)\frac{1}{K}+a_0\right](G(b_1)+D)-(b_1-b_0)\right\}K}{(1+a_0K)+(1+a_1K)[(2-\beta)+a_0K]}. \end{aligned} \tag{219}$$

Plugging Eqs. (217), (218), and (219) in Eq. (214) we deduce

$$\begin{aligned} \pi_1^c - \pi_1^* &= \left(\tfrac{1}{K}+\tfrac{1}{2}a_1\right){q_1^c}^2 - \left(v_1^*+\tfrac{1}{2}a_1\right){q_1^*}^2 \\ &= \left(\tfrac{1}{K}+\tfrac{1}{2}a_1\right)\left(\frac{\left\{\left[(1-\beta)\frac{1}{K}+a_0\right](G(b_1)+D)-(b_1-b_0)\right\}K}{(1+a_0K)+(1+a_1K)[(2-\beta)+a_0K]}\right)^2 \\ &\quad -\left(\frac{\frac{1}{2}a_1+\left(1+\frac{1}{2}a_1K\right)\left[(1-\beta)v_0^*+a_0\right]}{1+\left[(1-\beta)v_0^*+a_0\right]K}\right)\cdot \\ &\quad \cdot\left(\frac{\left[(1-\beta)v_0^*+a_0\right](G(b_1)+D)-(b_1-b_0)}{\left(v_0^*+a_0+a_1\right)+(1+a_1K)\left[(1-\beta)v_0^*+a_0\right]}\right)^2 \\ &= \tfrac{1}{2}K(2+a_1K)\left(\frac{\left[(1-\beta)\frac{1}{K}+a_0\right](G(b_1)+D)-(b_1-b_0)}{(1+a_0K)+(1+a_1K)[(2-\beta)+a_0K]}\right)^2 \\ &\quad -\tfrac{1}{2}\left(\frac{a_1+(2+a_1K)\left[(1-\beta)v_0^*+a_0\right]}{1+\left[(1-\beta)v_0^*+a_0\right]K}\right)\cdot \\ &\quad \cdot\left(\frac{\left[(1-\beta)v_0^*+a_0\right](G(b_1)+D)-(b_1-b_0)}{\left(v_0^*+a_0+a_1\right)+(1+a_1K)\left[(1-\beta)v_0^*+a_0\right]}\right)^2. \end{aligned} \tag{220}$$

Then, to prove the inequalities Eqs. (33) and (34) the following conditions has to be met:

$$\pi_1^c(1)-\pi_1^*(1) = (\pi_1^c-\pi_1^*)(1) < 0 \tag{221}$$

and

$$\lim_{\beta\to 0}\pi_1^c(\beta)-\lim_{\beta\to 0}\pi_1^*(\beta)=\lim_{\beta\to 0}(\pi_1^c-\pi_1^*)(\beta)>0. \tag{222}$$

Evaluating the expression of v_0^* given by Eq. (69) for $\beta = 1$ and using the notation $\overline{v_0^*} = v_0^*(1)$, one has

$$\begin{aligned} \overline{v_0^*} &= v_0^*(1) \\ &= \frac{2(a_0+a_1+a_0a_1K)}{\left(1+2a_0K+a_1K+a_0a_1K^2\right)+\sqrt{\left(1+2a_0K+a_1K+a_0a_1K^2\right)^2}} \\ &= \frac{a_0+a_1+a_0a_1K}{1+2a_0K+a_1K+a_0a_1K^2}. \end{aligned} \tag{223}$$

Now, we evaluate Eq. (220) for $\beta = 1$ to obtain

$$\begin{aligned} (\pi_1^c-\pi_1^*)(1) &= \tfrac{1}{2}K(2+a_1K)\left(\frac{[a_0](G(b_1)+D)-(b_1-b_0)}{(1+a_0K)+(1+a_1K)[1+a_0K]}\right)^2 \\ &\quad -\tfrac{1}{2}\left(\frac{a_1+(2+a_1K)[a_0]}{1+[a_0]K}\right)\left(\frac{[a_0](G(b_1)+D)-(b_1-b_0)}{\left(\overline{v_0^*}+a_0+a_1\right)+(1+a_1K)[a_0]}\right)^2 \\ &= \tfrac{1}{2}\frac{\left[a_0(G(b_1)+D)-(b_1-b_0)\right]^2}{1+a_0K}\frac{K}{2+2a_0K+a_1K+a_0a_1K^2} \\ &\quad -\tfrac{1}{2}\frac{\left[a_0(G(b_1)+D)-(b_1-b_0)\right]^2}{1+a_0K}\frac{2a_0+a_1+a_0a_1K}{\left(\overline{v_0^*}+2a_0+a_1+a_0a_1K\right)^2} \\ &= U_1\frac{V_1}{W_1}, \end{aligned} \tag{224}$$

where

$$U_1 = \tfrac{1}{2}\frac{\left[a_0(G(b_1)+D)-(b_1-b_0)\right]^2}{1+a_0K}, \tag{225}$$

$$\begin{aligned} V_1 &= K\left(\overline{v_0^*}+2a_0+a_1+a_0a_1K\right)^2 \\ &\quad -(2a_0+a_1+a_0a_1K)\left(2+2a_0K+a_1K+a_0a_1K^2\right) \end{aligned} \tag{226}$$

and

$$W_1 = \left(2+2a_0K+a_1K+a_0a_1K^2\right)\left(\overline{v_0^*}+2a_0+a_1+a_0a_1K\right)^2. \tag{227}$$

Given the values of $a_0, a_1, \overline{v_0^*}$, and K, it isn't difficult to see that $U_1 > 0$ and $W_1 > 0$. Hence, to prove Eq. (221) it is enough to show that $V_1 < 0$. Indeed, plugging the expression of $\overline{v_0^*}$ given by Eq. (223) in Eq. (226), we have that

$$\begin{aligned} V_1 &= K\left(\overline{v_0^*}+2a_0+a_1+a_0a_1K\right)^2 \\ &\quad -(2a_0+a_1+a_0a_1K)\left(2+2a_0K+a_1K+a_0a_1K^2\right) \\ &= K\left(\frac{a_0+a_1+a_0a_1K}{1+2a_0K+a_1K+a_0a_1K^2}+2a_0+a_1+a_0a_1K\right)^2 \\ &\quad -(2a_0+a_1+a_0a_1K)\left(2+2a_0K+a_1K+a_0a_1K^2\right) \\ &= \frac{K\left[a_0+a_1+a_0a_1K+(2a_0+a_1+a_0a_1K)\left(1+2a_0K+a_1K+a_0a_1K^2\right)\right]^2}{\left(1+2a_0K+a_1K+a_0a_1K^2\right)^2} \\ &\quad -\frac{(2a_0+a_1+a_0a_1K)\left(2+2a_0K+a_1K+a_0a_1K^2\right)}{\left(1+2a_0K+a_1K+a_0a_1K^2\right)^2}\cdot \\ &\quad \cdot\left(1+2a_0K+a_1K+a_0a_1K^2\right)^2 \\ &= \frac{P}{Q}, \end{aligned} \tag{228}$$

where

$$\begin{aligned} P &= K\big[a_0+a_1+a_0a_1K+ \\ &\quad +(2a_0+a_1+a_0a_1K)\left(1+2a_0K+a_1K+a_0a_1K^2\right)\big]^2 \\ &\quad -(2a_0+a_1+a_0a_1K)\left(2+2a_0K+a_1K+a_0a_1K^2\right)\cdot \\ &\quad \cdot\left(1+2a_0K+a_1K+a_0a_1K^2\right)^2, \end{aligned} \tag{229}$$

and

$$Q = \left(1+2a_0K+a_1K+a_0a_1K^2\right)^2. \tag{230}$$

For any fixed values of a_0, a_1, and K, it is clear that $Q > 0$, and

$$\begin{aligned} P &< K\big[(2a_0+a_1+a_0a_1K) \\ &\quad +(2a_0+a_1+a_0a_1K)\left(1+2a_0K+a_1K+a_0a_1K^2\right)\big]^2 \\ &\quad -(2a_0+a_1+a_0a_1K)\left(2+2a_0K+a_1K+a_0a_1K^2\right)\cdot \\ &\quad \cdot\left(1+2a_0K+a_1K+a_0a_1K^2\right)^2 \\ &= -(2a_0+a_1+a_0a_1K)\left(2+2a_0K+a_1K+a_0a_1K^2\right)<0. \end{aligned} \tag{231}$$

Therefore, $P < 0$, which shows that

$$V_1 = \frac{P}{Q} < 0. \tag{232}$$

Then since $U_1 > 0$ and $W_1 > 0$, we have that

$$(\pi_1^c - \pi_1^*)(1) = U_1\frac{V_1}{W_1} < 0, \tag{233}$$

which proves Eq. (221).

Now, we need only to prove Eq. (222). Using the notation $\widehat{v_0^*} = \lim_{\beta\to 0} v_0^*(\beta)$ given by Eq. (171), from Eq. (220) we have that

$$\begin{aligned}
&\lim_{\beta\to 0}(\pi_1^c-\pi_1^*)(\beta)\\
&=\tfrac{1}{2}K(2+a_1K)\left(\frac{\left[\frac{1}{K}+a_0\right](G(b_1)+D)-(b_1-b_0)}{(1+a_0K)+(1+a_1K)[2+a_0K]}\right)^2\\
&\quad-\tfrac{1}{2}\left(\frac{a_1+(2+a_1K)\left[\widehat{v_0^*}+a_0\right]}{1+\left[\widehat{v_0^*}+a_0\right]K}\right)\left(\frac{\left[\widehat{v_0^*}+a_0\right](G(b_1)+D)-(b_1-b_0)}{\left(\widehat{v_0^*}+a_0+a_1\right)+(1+a_1K)\left[\widehat{v_0^*}+a_0\right]}\right)^2\\
&=\tfrac{1}{2}K(2+a_1K)\left(\frac{\left(\frac{1}{K}+a_0\right)(G(b_1)+D)-(b_1-b_0)}{(1+a_1K)+(1+a_0K)(2+a_1K)}\right)^2\\
&\quad-\tfrac{1}{2}\frac{1}{1+\left(\widehat{v_0^*}+a_0\right)K}\frac{\left[\left(\widehat{v_0^*}+a_0\right)(G(b_1)+D)-(b_1-b_0)\right]^2}{a_1+(2+a_1K)\left(\widehat{v_0^*}+a_0\right)}\\
&=\tfrac{1}{2}\frac{V_2}{W_2},
\end{aligned}\tag{234}$$

where

$$\begin{aligned}
V_2=&K(2+a_1K)\left[\left(\tfrac{1}{K}+a_0\right)(G(b_1)+D)-(b_1-b_0)\right]^2\cdot\\
&\cdot\left[1+\left(\widehat{v_0^*}+a_0\right)K\right]\left[a_1+(2+a_1K)\left(\widehat{v_0^*}+a_0\right)\right]\\
&-[(1+a_1K)+(1+a_0K)(2+a_1K)]^2\cdot\\
&\cdot\left[\left(\widehat{v_0^*}+a_0\right)(G(b_1)+D)-(b_1-b_0)\right]^2,
\end{aligned}\tag{235}$$

and

$$\begin{aligned}
W_2=&[(1+a_1K)+(1+a_0K)(2+a_1K)]^2\cdot\\
&\cdot\left[1+\left(\widehat{v_0^*}+a_0\right)K\right]\left[a_1+(2+a_1K)\left(\widehat{v_0^*}+a_0\right)\right].
\end{aligned}\tag{236}$$

For arbitrary fixed values of $a_0, a_1, \widehat{v_0^*}$, and K, it is evident that $W_2>0$. Hence, to prove Eq. (222) it lacks only to show that $V_2>0$. Indeed,

$$\begin{aligned}
V_2=&\left[(1+a_1K)+\left(\widehat{v_0^*}+a_0\right)(2+a_1K)K\right]^2\cdot\\
&\cdot\left[\left(\tfrac{1}{K}+a_0\right)(G(b_1)+D)-(b_1-b_0)\right]^2\\
&-\left[(1+a_1K)+\left(\tfrac{1}{K}+a_0\right)(2+a_1K)K\right]^2\cdot\\
&\cdot\left[\left(\widehat{v_0^*}+a_0\right)(G(b_1)+D)-(b_1-b_0)\right]^2\\
&-\left[\left(\tfrac{1}{K}+a_0\right)(G(b_1)+D)-(b_1-b_0)\right]^2.
\end{aligned}\tag{237}$$

Now introduce the following notation:

$$\eta=1+a_1K>0,\tag{238}$$

$$\xi=K(1+\eta)=K(2+a_1K)>0,\tag{239}$$

$$\begin{aligned}\mathscr{Z}=&\eta+a_0\xi=(1+a_1K)\\&+a_0K(2+a_1K)>0,\end{aligned}\tag{240}$$

$$G1=G(b_1)+D>0\tag{241}$$

and

$$\begin{aligned}G3=&a_0G1-(b_1-b_0)=\\&=a_0(G(b_1)+D)-(b_1-b_0)>0.\end{aligned}\tag{242}$$

Based on that, we can rewrite Eq. (237) as follows:

$$\begin{aligned}
V_2=&\left(\widehat{v_0^*}\xi+\mathscr{Z}\right)^2\left(\tfrac{1}{K}G1+G3\right)^2\\
&-\left(\tfrac{1}{K}\xi+\mathscr{Z}\right)^2\left(\widehat{v_0^*}G1+G3\right)^2-\left(\tfrac{1}{K}G1+G3\right)^2\\
=&\left(\tfrac{1}{K^2}-\widehat{v_0^*}^2\right)\left(\mathscr{Z}^2G1^2-\xi^2G3^2\right)-\tfrac{1}{K}G1\left(\tfrac{1}{K}G1+2G3\right)\\
&+2\mathscr{Z}G3\left(\tfrac{1}{K}-\widehat{v_0^*}\right)(\mathscr{Z}G1-\xi G3)-G3^2\\
&+2\tfrac{1}{K}\widehat{v_0^*}\xi G1\left(\tfrac{1}{K}-\widehat{v_0^*}\right)(\mathscr{Z}G1-\xi G3)\\
=&\mathscr{P}_1+\mathscr{Q}_1+\mathscr{R}_1,
\end{aligned}\tag{243}$$

where

$$\mathscr{P}_1=\left(\tfrac{1}{K^2}-\widehat{v_0^*}^2\right)\left(\mathscr{Z}^2G1^2-\xi^2G3^2\right)-\tfrac{1}{K}G1\left(\tfrac{1}{K}G1+2G3\right),\tag{244}$$

$$\mathscr{Q}_1=2\mathscr{Z}G3\left(\tfrac{1}{K}-\widehat{v_0^*}\right)(\mathscr{Z}G1-\xi G3)-G3^2\tag{245}$$

and

$$\mathscr{R}_1=2\tfrac{1}{K}\widehat{v_0^*}\xi G1\left(\tfrac{1}{K}-\widehat{v_0^*}\right)(\mathscr{Z}G1-\xi G3).\tag{246}$$

Now, we are going to show that

$$\mathscr{Z}G1-\xi G3>0.\tag{247}$$

Using Eqs. (240) and (242), we have that

$$\begin{aligned}\mathscr{Z}G1-\xi G3=&(\eta+a_0\xi)G1-\xi(a_0G1-(b_1-b_0))\\=&\eta G1+\xi(b_1-b_0)\geq\eta G1>0,\end{aligned}\tag{248}$$

which proves Eq. (247).
Thus, given the values of $\widehat{v_0^*}, K$, Eqs. (239), (241), (180), and (247), we can conclude that $\mathscr{R}_1>0$.
Now,

$$\begin{aligned}\mathscr{Q}_1=&2\mathscr{Z}G3\left(\tfrac{1}{K}-\widehat{v_0^*}\right)(\mathscr{Z}G1-\xi G3)-G3^2\\=&\left[2\mathscr{Z}\left(\tfrac{1}{K}-\widehat{v_0^*}\right)(\mathscr{Z}G1-\xi G3)-G3\right]G3,\end{aligned}\tag{249}$$

and using Eq. (242) we can rewrite Eq. (249) as follows:

$$\begin{aligned}
\mathscr{Q}_1=&\left\{2\mathscr{Z}\left(\tfrac{1}{K}-\widehat{v_0^*}\right)(\mathscr{Z}G1-\xi[a_0G1-(b_1-b_0)])\right.\\
&\left.-[a_0G1-(b_1-b_0)]\right\}G3\\
=&\left\{\left[2\mathscr{Z}(\mathscr{Z}-a_0\xi)\left(\tfrac{1}{K}-\widehat{v_0^*}\right)-a_0\right]G1\right.\\
&\left.+\left[2\xi\mathscr{Z}\left(\tfrac{1}{K}-\widehat{v_0^*}\right)+1\right](b_1-b_0)\right\}G3.
\end{aligned}\tag{250}$$

Moreover, from Eq. (240), we have that

$$\eta=\mathscr{Z}-a_0\xi=1+a_1K.\tag{251}$$

Substituting Eq. (251) in Eq. (250) we have that

$$\begin{aligned}
\mathscr{Q}_1=&\left\{\left[2\mathscr{Z}(1+a_1K)\left(\tfrac{1}{K}-\widehat{v_0^*}\right)-a_0\right]G1\right.\\
&\left.+\left[2\xi\mathscr{Z}\left(\tfrac{1}{K}-\widehat{v_0^*}\right)+1\right](b_1-b_0)\right\}G3\\
=&\left\{\left[2a_1K\mathscr{Z}\left(\tfrac{1}{K}-\widehat{v_0^*}\right)+2\mathscr{Z}\left(\tfrac{1}{K}-\widehat{v_0^*}\right)-a_0\right]G1\right.\\
&\left.+\left[2\xi\mathscr{Z}\left(\tfrac{1}{K}-\widehat{v_0^*}\right)+1\right](b_1-b_0)\right\}G3\\
=&[(V_3+W_3)G1+U_3(b_1-b_0)]G3,
\end{aligned}\tag{252}$$

where,

$$V_3=2a_1K\mathscr{Z}\left(\frac{1}{K}-\widehat{v_0^*}\right),\tag{253}$$

$$W_3 = 2\mathscr{L}\left(\frac{1}{K} - \widehat{v_0^*}\right) - a_0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (254)$$

and

$$U_3 = 2\xi\mathscr{L}\left(\frac{1}{K} - \widehat{v_0^*}\right) + 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (255)$$

For any given values of a_1, K, ξ ,and $\mathscr{L}$, one easily deduces that $V_3 > 0$ and $U_3 > 0$. Now, we are going to show that $W_3 > 0$. In order to do that, we first substitute Eq. (240) in Eq. (254) to get:

$$\begin{aligned} W_3 &= 2[(1+a_1K)+a_0K(2+a_1K)]\left(\tfrac{1}{K}-\widehat{v_0^*}\right)-a_0 \\ &> a_0K(2+a_1K)\left(\tfrac{1}{K}-\widehat{v_0^*}\right)-a_0 \\ &= a_0\left[K(2+a_1K)\left(\tfrac{1}{K}-\widehat{v_0^*}\right)-1\right]. \end{aligned} \quad . \quad . \quad . \quad . \quad (256)$$

Now, making use of the expression of $\widehat{v_0^*}$ given by Eq. (171) we have that

$$\begin{aligned} &\tfrac{1}{K}-\widehat{v_0^*} \\ &= \tfrac{1}{K} - \frac{2(a_0+a_1+a_0a_1K)}{\left(2a_0K+a_0a_1K^2\right)+\sqrt{\left(2a_0K+a_0a_1K^2\right)^2+4\left(2K+a_1K^2\right)(a_0+a_1+a_0a_1K)}} \\ &= \frac{\sqrt{\left(2a_0K+a_0a_1K^2\right)^2+4\left(2K+a_1K^2\right)(a_0+a_1+a_0a_1K)}-a_1K(2+a_0K)}{K\left[\left(2a_0K+a_0a_1K^2\right)+\sqrt{\left(2a_0K+a_0a_1K^2\right)^2+4\left(2K+a_1K^2\right)(a_0+a_1+a_0a_1K)}\right]}. \end{aligned}$$
$$. \quad (257)$$

Now plugging Eq. (257) in Eq. (256) we get:

$$\begin{aligned} W_3 &> a_0\left[K(2+a_1K)\left(\frac{1}{K}-\widehat{v_0^*}\right)-1\right] \\ &= a_0\frac{V_4}{W_4}, \end{aligned} \quad . \quad . \quad (258)$$

where

$$\begin{aligned} V_4 &= -K(2+a_1K)[a_0(1+a_1K)+2a_1]+ \\ &(1+a_1K)\sqrt{\left(2a_0K+a_0a_1K^2\right)^2+4\left(2K+a_1K^2\right)(a_0+a_1+a_0a_1K)} \end{aligned} \quad (259)$$

and

$$\begin{aligned} W_4 &= \left(2a_0K+a_0a_1K^2\right)+ \\ &\sqrt{\left(2a_0K+a_0a_1K^2\right)^2+4\left(2K+a_1K^2\right)(a_0+a_1+a_0a_1K)}. \end{aligned} \quad (260)$$

For any values of a_0, a_1, and K, we see that $W_4 > 0$; thus, to compute the value of Eq. (258) we need only to estimate the value of V_4. Suppose that

$$V_4 \leq 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (261)$$

Then, we would have

$$\begin{aligned} &(1+a_1K)\sqrt{\left(2a_0K+a_0a_1K^2\right)^2+4\left(2K+a_1K^2\right)(a_0+a_1+a_0a_1K)} \\ &\leq K(2+a_1K)[a_0(1+a_1K)+2a_1]. \end{aligned} \quad (262)$$

Both sides of Eq. (262) are positive, then, squaring them we have that

$$\begin{aligned} &(1+a_1K)^2\left[\left(2a_0K+a_0a_1K^2\right)^2+4\left(2K+a_1K^2\right)(a_0+a_1+a_0a_1K)\right] \\ &\leq K^2(2+a_1K)^2[a_0(1+a_1K)+2a_1]^2. \end{aligned} \quad (263)$$

Thus,

$$\begin{aligned} &(1+a_1K)^2\left[a_0^2K^2(2+a_1K)^2+4K(2+a_1K)(a_0+a_1+a_0a_1K)\right] \\ &\leq K^2(2+a_1K)^2[a_0(1+a_1K)+2a_1]^2, \end{aligned} \quad (264)$$

which leads to

$$\begin{aligned} &K(2+a_1K)(1+a_1K)^2\left[a_0^2K(2+a_1K)+4(a_0+a_1+a_0a_1K)\right] \\ &\leq K^2(2+a_1K)^2[a_0(1+a_1K)+2a_1]^2, \end{aligned} \quad . \quad (265)$$

where $K(2+a_1K) > 0$. Therefore,

$$\begin{aligned} &(1+a_1K)^2\left[a_0^2K(2+a_1K)+4(a_0+a_1+a_0a_1K)\right] \\ &\leq K(2+a_1K)[a_0(1+a_1K)+2a_1]^2. \end{aligned} \quad . \quad . \quad . \quad . \quad (266)$$

Now expanding the squares we arrive at:

$$\begin{aligned} &(1+a_1K)^2\left[a_0^2K(2+a_1K)+4(a_0+a_1+a_0a_1K)\right] \\ &\leq K(2+a_1K)\left[a_0^2(1+a_1K)^2+4a_0a_1(1+a_1K)+4a_1^2\right], \end{aligned} \quad . \quad . \quad (267)$$

and by distributing some terms:

$$\begin{aligned} &a_0^2K(2+a_1K)(1+a_1K)^2+4(a_0+a_1+a_0a_1K)(1+a_1K)^2 \\ &\leq a_0^2K(2+a_1K)(1+a_1K)^2 \\ &\quad +4a_0a_1K(2+a_1K)(1+a_1K)+4a_1^2K(2+a_1K). \end{aligned} \quad . \quad (268)$$

Thus

$$\begin{aligned} &4(a_0+a_1+a_0a_1K)(1+a_1K)^2 \\ &\leq 4a_0a_1K(2+a_1K)(1+a_1K)+4a_1^2K(2+a_1K), \end{aligned} \quad . \quad . \quad . \quad . \quad (269)$$

and distributing again:

$$\begin{aligned} &4(a_0+a_1)(1+a_1K)^2+4a_0a_1K(1+a_1K)^2 \\ &\leq 4a_0a_1K(1+a_1K)+4a_0a_1K(1+a_1K)^2+4a_1^2K(2+a_1K), \end{aligned} \quad . \quad (270)$$

which leads to

$$\begin{aligned} &4(a_0+a_1)(1+a_1K)^2 \\ &\leq 4a_0a_1K(1+a_1K)+4a_1^2K(2+a_1K). \end{aligned} \quad . \quad . \quad (271)$$

Finally,

$$\begin{aligned} &(a_0+a_1)(1+a_1K)^2 \\ &\leq a_0a_1K(1+a_1K)+a_1^2K(2+a_1K), \end{aligned} \quad . \quad . \quad . \quad (272)$$

and by distributing some terms and expanding he squares again we deduce that

$$\begin{aligned} &(a_0+a_1)\left(1+2a_1K+a_1^2K^2\right) \\ &\leq a_0a_1K(1+a_1K)+a_1^2K(1+a_1K)+a_1^2K, \end{aligned} \quad (273)$$

thus,

$$\begin{aligned} &(a_0+a_1)\left(1+2a_1K+a_1^2K^2\right) \\ &\leq (a_0+a_1)a_1K(1+a_1K)+a_1^2K, \end{aligned} \quad . \quad . \quad . \quad . \quad (274)$$

which leads to

$$\begin{aligned} &(a_0+a_1)\left(1+2a_1K+a_1^2K^2\right) \\ &\leq (a_0+a_1)\left(a_1K+a_1^2K^2\right)+a_1^2K. \end{aligned} \quad . \quad . \quad . \quad . \quad (275)$$

Hence,

$$(a_0+a_1)(1+a_1K) \leq a_1^2K, \quad . \quad . \quad . \quad . \quad . \quad . \quad (276)$$

which finally leads to

$$a_0+a_1+a_0a_1K+a_1^2K \le a_1^2K, \quad . \; . \; . \; . \tag{277}$$

which is impossible since $a_0+a_1+a_0a_1K>0$.
On a base of that, we conclude that

$$V_4>0, \tag{278}$$

so

$$W_3>a_0\frac{V_4}{W_4}>0. \tag{279}$$

Now, since $V_3>0$, $W_3>0$, $U_3>0$, and given the values of $b_0, b_1, G1$ an $G3$, we have that

$$\mathscr{Q}_1=[(V_3+W_3)\,G1+U_3(b_1-b_0)]\,G3>0. \tag{280}$$

So we need only to estimate the value of $\mathscr{P}_1$:

$$\begin{aligned}\mathscr{P}_1&=\left(\tfrac{1}{K^2}-\widehat{v_0^*}^2\right)\left(\mathscr{Z}^2G1^2-\xi^2G3^2\right)-\tfrac{1}{K}G1\left(\tfrac{1}{K}G1+2G3\right)\\&=\left(\tfrac{1}{K}\mathscr{Z}G1+\tfrac{1}{K}\xi G3+\widehat{v_0^*}\mathscr{Z}G1+\widehat{v_0^*}\xi G3\right)\left(\tfrac{1}{K}-\widehat{v_0^*}\right)\cdot\\&\quad\cdot(\mathscr{Z}G1-\xi G3)-\tfrac{1}{K}G1\left(\tfrac{1}{K}G1+2G3\right).\end{aligned} \tag{281}$$

By Eqs. (239), (242), and (251), we have that

$$\xi=K\,(2+a_1K)>2K, \tag{282}$$

$$\begin{aligned}\mathscr{Z}G1-\xi G3&=\mathscr{Z}G1-\xi\,[a_0G1-(b_1-b_0)]\\&\ge\mathscr{Z}G1-a_0\xi G1=\eta G1,\end{aligned} \tag{283}$$

$$\mathscr{Z}=\eta+a_0\xi>\eta, \tag{284}$$

and

$$\widehat{v_0^*}\xi G3>0. \tag{285}$$

Using inequalities Eqs. (282)–(285) in Eq. (281) we get:

$$\begin{aligned}\mathscr{P}_1&>\left(\tfrac{1}{K}\mathscr{Z}G1+2G3+\widehat{v_0^*}\eta G1\right)\left(\tfrac{1}{K}-\widehat{v_0^*}\right)\eta G1\\&\quad-\tfrac{1}{K}G1\left(\tfrac{1}{K}G1+2G3\right)\\&=\Big\{\Big[\left(\tfrac{1}{K}\mathscr{Z}+\widehat{v_0^*}\eta\right)\eta G1+2\eta G3\Big]\left(\tfrac{1}{K}-\widehat{v_0^*}\right)\\&\quad-\tfrac{1}{K}\left(\tfrac{1}{K}G1+2G3\right)\Big\}G1.\end{aligned} \tag{286}$$

Now making use of Eqs. (238), (240), and (282), we come to

$$\eta=1+a_1K>1 \tag{287}$$

and

$$\mathscr{Z}=\eta+a_0\xi>\eta+2a_0K. \tag{288}$$

Then, applying inequalities Eqs. (287) and (288) to Eq. (286) one gets:

$$\begin{aligned}\mathscr{P}_1&>\Big\{\Big[\left(\tfrac{1}{K}\mathscr{Z}+\widehat{v_0^*}\eta\right)\eta G1+2\eta G3\Big]\left(\tfrac{1}{K}-\widehat{v_0^*}\right)\\&\quad-\tfrac{1}{K}\left(\tfrac{1}{K}G1+2G3\right)\Big\}G1\\&>\Big\{\Big[\left(\tfrac{1}{K}(\eta+2a_0K)+\widehat{v_0^*}\right)G1+2\eta G3\Big]\left(\tfrac{1}{K}-\widehat{v_0^*}\right)\\&\quad-\tfrac{1}{K}\left(\tfrac{1}{K}G1+2G3\right)\Big\}G1\\&=\Big\{\Big[\left(\tfrac{1}{K}\eta+2a_0+\widehat{v_0^*}\right)G1+2\eta G3\Big]\left(\tfrac{1}{K}-\widehat{v_0^*}\right)\\&\quad-\tfrac{1}{K}\left(\tfrac{1}{K}G1+2G3\right)\Big\}G1.\end{aligned} \tag{289}$$

Substituting the value of $G3$ given by Eq. (242) in Eq. (289) we have:

$$\begin{aligned}\mathscr{P}_1&>\Big\{\Big[\left(\tfrac{1}{K}\eta+2a_0+\widehat{v_0^*}\right)G1+2\eta G3\Big]\left(\tfrac{1}{K}-\widehat{v_0^*}\right)\\&\quad-\tfrac{1}{K}\left(\tfrac{1}{K}G1+2G3\right)\Big\}G1\\&=\Big\{\Big[\left(\tfrac{1}{K}\eta+2a_0+\widehat{v_0^*}\right)G1+2\eta\,[a_0G1-(b_1-b_0)]\Big]\left(\tfrac{1}{K}-\widehat{v_0^*}\right)\\&\quad-\tfrac{1}{K}\Big\{\tfrac{1}{K}G1+2[a_0G1-(b_1-b_0)]\Big\}\Big\}G1\\&=\Big\{\Big[\left(\tfrac{1}{K}\eta+2a_0+2a_0\eta+\widehat{v_0^*}\right)\left(\tfrac{1}{K}-\widehat{v_0^*}\right)-\tfrac{1}{K}\left(\tfrac{1}{K}+2a_0\right)\Big]G1\\&\quad+2\Big[\tfrac{1}{K}-\eta\left(\tfrac{1}{K}-\widehat{v_0^*}\right)\Big](b_1-b_0)\Big\}G1.\end{aligned} \tag{290}$$

Next, we substitute the value of η given by Eq. (238) in Eq. (290) to obtain:

$$\begin{aligned}\mathscr{P}_1&>\Big\{\Big[\left(\tfrac{1}{K}\eta+2a_0+2a_0\eta+\widehat{v_0^*}\right)\left(\tfrac{1}{K}-\widehat{v_0^*}\right)-\tfrac{1}{K}\left(\tfrac{1}{K}+2a_0\right)\Big]G1\\&\quad+2\Big[\tfrac{1}{K}-\eta\left(\tfrac{1}{K}-\widehat{v_0^*}\right)\Big](b_1-b_0)\Big\}G1\\&=\Big\{\Big[\left(\tfrac{1}{K}(1+a_1K)+2a_0+2a_0\eta+\widehat{v_0^*}\right)\left(\tfrac{1}{K}-\widehat{v_0^*}\right)\\&\quad-\tfrac{1}{K}\left(\tfrac{1}{K}+2a_0\right)\Big]G1+2\Big[\tfrac{1}{K}-(1+a_1K)\left(\tfrac{1}{K}-\widehat{v_0^*}\right)\Big](b_1-b_0)\Big\}G1\\&=\Big\{\Big[(a_1+2a_0\eta)\left(\tfrac{1}{K}-\widehat{v_0^*}\right)-\widehat{v_0^*}\left(2a_0+\widehat{v_0^*}\right)\Big]G1\\&\quad+2\Big[(1+a_1K)\widehat{v_0^*}-a_1\Big](b_1-b_0)\Big\}G1\\&=[V_5G1+2W_5(b_1-b_0)]G1,\end{aligned} \tag{291}$$

where

$$V_5=(a_1+2a_0\eta)\left(\frac{1}{K}-\widehat{v_0^*}\right)-\widehat{v_0^*}\left(2a_0+\widehat{v_0^*}\right) \tag{292}$$

and

$$W_5=(1+a_1K)\,\widehat{v_0^*}-a_1. \tag{293}$$

Now, we only need to show that $V_5>0$ and $W_5>0$.

First, since $v_0^*(\beta)$ is strictly decreasing, we have $\widehat{v_0^*}=\lim\limits_{\beta\to 0}v_0^*(\beta)>v_0^*(1)=\overline{v_0^*}$, thus,

$$W_5=(1+a_1K)\,\widehat{v_0^*}-a_1>(1+a_1K)\,\overline{v_0^*}-a_1. \tag{294}$$

Substituting the value of $\overline{v_0^*}$ given by Eq. (223) in Eq. (294) we have that:

$$\begin{aligned}W_5&>(1+a_1K)\overline{v_0^*}-a_1\\&=(1+a_1K)\left(\frac{a_0+a_1+a_0a_1K}{1+2a_0K+a_1K+a_0a_1K^2}\right)-a_1\\&=\frac{(1+a_1K)(a_0+a_1+a_0a_1K)-a_1\left(1+2a_0K+a_1K+a_0a_1K^2\right)}{1+2a_0K+a_1K+a_0a_1K^2}\\&=\frac{V_6}{W_6},\end{aligned} \tag{295}$$

where

$$\begin{aligned}V_6=&\,(1+a_1K)\,(a_0+a_1+a_0a_1K)\\&-a_1\left(1+2a_0K+a_1K+a_0a_1K^2\right)\end{aligned} \tag{296}$$

and

$$W_6=1+2a_0K+a_1K+a_0a_1K^2. \tag{297}$$

Given the values of a_0, a_1 and K, it's easy to see that $W_6>$

0. Now we calculate the value of V_6:

$$\begin{aligned} V_6 =& (1+a_1K)(a_0+a_1+a_0a_1K) \\ &- a_1\left(1+2a_0K+a_1K+a_0a_1K^2\right) \\ =& a_0 > 0. \end{aligned} \quad (298)$$

Therefore, $V_6 > 0$ and

$$W_5 > \frac{V_6}{W_6} > 0. \quad (299)$$

So we only lack showing that $V_5 > 0$:

$$\begin{aligned} V_5 =& (a_1+2a_0\eta)\left(\tfrac{1}{K}-\widehat{v_0^*}\right)-\widehat{v_0^*}\left(2a_0+\widehat{v_0^*}\right) \\ =& \left[(a_1+a_0\eta)\left(\tfrac{1}{K}-\widehat{v_0^*}\right)-\widehat{v_0^*}\left(a_0+\widehat{v_0^*}\right)\right]+a_0\left[\eta\left(\tfrac{1}{K}-\widehat{v_0^*}\right)-\widehat{v_0^*}\right] \\ =& V_7+a_0W_7, \end{aligned} \quad (300)$$

where

$$V_7 = (a_1+a_0\eta)\left(\frac{1}{K}-\widehat{v_0^*}\right)-\widehat{v_0^*}\left(a_0+\widehat{v_0^*}\right) \quad (301)$$

and

$$W_7 = \eta\left(\frac{1}{K}-\widehat{v_0^*}\right)-\widehat{v_0^*}. \quad (302)$$

Finally, we will demonstrate that $V_7 > 0$ and $W_7 > 0$. Substituting the value of η given by Eq. (238) in V_7 we get:

$$\begin{aligned} V_7 =& (a_1+a_0\eta)\left(\tfrac{1}{K}-\widehat{v_0^*}\right)-\widehat{v_0^*}\left(a_0+\widehat{v_0^*}\right) \\ =& [a_1+a_0(1+a_1K)]\left(\tfrac{1}{K}-\widehat{v_0^*}\right)-\widehat{v_0^*}\left(a_0+\widehat{v_0^*}\right) \\ =& \tfrac{1}{K}(a_1+a_0+a_0a_1K)-\widehat{v_0^*}(a_1+2a_0+a_0a_1K)-\widehat{v_0^*}^2. \end{aligned} \quad (303)$$

Now, we use the relationship

$$\begin{aligned} &(1-\beta)\left(-2\tau+a_1\tau^2\right)v_0^2 \\ &+\left(\beta-2a_0\tau-\beta a_1\tau+a_0a_1\tau^2\right)v_0 \\ &-(a_0+a_1-a_0a_1\tau)=0, \end{aligned}$$

given by Eq. (44), for $\tau = -K$. Now by applying the limit when $\beta \to 0$, one obtains the following equality:

$$\left(2K+a_1K^2\right)\widehat{v_0^*}^2+\left(2a_0K+a_0a_1K^2\right)\widehat{v_0^*}-(a_0+a_1+a_0a_1K)=0. \quad (304)$$

Therefore,

$$\begin{aligned} &\left(2K+a_1K^2\right)\widehat{v_0^*}^2+\left(2a_0K+a_0a_1K^2\right)\widehat{v_0^*}-(a_0+a_1+a_0a_1K) \\ &=\Big[\widehat{v_0^*}^2+(a_1+2a_0+a_0a_1K)\widehat{v_0^*}-\tfrac{1}{K}(a_0+a_1+a_0a_1K) \\ &\quad +(1+a_1K)\widehat{v_0^*}^2-a_1\widehat{v_0^*}\Big]K=0. \end{aligned} \quad (305)$$

Since $K > 0$, we have that

$$\begin{aligned} &\widehat{v_0^*}^2+(a_1+2a_0+a_0a_1K)\widehat{v_0^*}-\tfrac{1}{K}(a_0+a_1+a_0a_1K) \\ &\quad +(1+a_1K)\widehat{v_0^*}^2-a_1\widehat{v_0^*}=0, \end{aligned} \quad (306)$$

thus,

$$\begin{aligned} &(1+a_1K)\widehat{v_0^*}^2-a_1\widehat{v_0^*}=\frac{1}{K}(a_0+a_1+a_0a_1K) \\ &-(a_1+2a_0+a_0a_1K)\widehat{v_0^*}-\widehat{v_0^*}^2. \end{aligned} \quad (307)$$

Applying equality Eq. (307) to Eq. (303) we have that:

$$\begin{aligned} V_7 =& \frac{1}{K}(a_0+a_1+a_0a_1K) \\ &-(a_1+2a_0+a_0a_1K)\widehat{v_0^*}-\widehat{v_0^*}^2 \\ =& (1+a_1K)\widehat{v_0^*}^2-a_1\widehat{v_0^*} \\ =& \left[(1+a_1K)\widehat{v_0^*}-a_1\right]\widehat{v_0^*} \\ =& W_5\widehat{v_0^*}. \end{aligned} \quad (308)$$

Now recalling that $W_5 > 0$ and $\widehat{v_0^*} > 0$, we have:

$$V_7 = W_5\widehat{v_0^*} > 0. \quad (309)$$

So we only need to show that $W_7 > 0$. To do this, we plug the value of η given by Eq. (238) in W_7 to get:

$$\begin{aligned} W_7 =& \eta\left(\frac{1}{K}-\widehat{v_0^*}\right)-\widehat{v_0^*} \\ =& (1+a_1K)\left(\frac{1}{K}-\widehat{v_0^*}\right)-\widehat{v_0^*} \\ =& \frac{1}{K}(1+a_1K)-(2+a_1K)\widehat{v_0^*}. \end{aligned} \quad (310)$$

Using relationship Eq. (304) we have that:

$$\begin{aligned} &\left(2K+a_1K^2\right)\widehat{v_0^*}^2+\left(2a_0K+a_0a_1K^2\right)\widehat{v_0^*}-(a_0+a_1+a_0a_1K) \\ &=\left\{(2+a_1K)\widehat{v_0^*}^2-\tfrac{1}{K}a_1+a_0\left[(2+a_1K)\widehat{v_0^*}-\tfrac{1}{K}(1+a_1K)\right]\right\}K. \end{aligned} \quad (311)$$

Since $K > 0$, we have that:

$$(2+a_1K)\widehat{v_0^*}^2-\tfrac{1}{K}a_1+a_0\left[(2+a_1K)\widehat{v_0^*}-\tfrac{1}{K}(1+a_1K)\right]=0, \quad (312)$$

which implies

$$(2+a_1K)\widehat{v_0^*}^2-\tfrac{1}{K}a_1=-a_0\left[(2+a_1K)\widehat{v_0^*}-\tfrac{1}{K}(1+a_1K)\right],. \quad (313)$$

As $a_0 > 0$, one gets:

$$\tfrac{1}{a_0}\left[(2+a_1K)\widehat{v_0^*}^2-\tfrac{1}{K}a_1\right]=\tfrac{1}{K}(1+a_1K)-(2+a_1K)\widehat{v_0^*}. \quad (314)$$

Now, applying equality Eq. (314) to Eq. (310), we deduce:

$$\begin{aligned} W_7 =& \frac{1}{K}(1+a_1K)-(2+a_1K)\widehat{v_0^*} \\ =& \frac{1}{a_0}\left[(2+a_1K)\widehat{v_0^*}^2-\frac{1}{K}a_1\right] \\ =& \frac{U_8}{a_0}, \end{aligned} \quad (315)$$

where

$$U_8 = (2+a_1K)\widehat{v_0^*}^2-\frac{1}{K}a_1. \quad (316)$$

Finally, let us suppose, on the contrary, that

$$U_8 \leq 0. \quad (317)$$

Substituting the value of $\widehat{v_0^*}$, given by Eq. (171), in U_8 we have:

$$\begin{aligned}U_8 &= (2+a_1K)\,\widehat{v_0^*}^2 - \frac{1}{K}a_1 \\ &= \frac{V_9}{W_9} \leq 0,\end{aligned} \tag{318}$$

where

$$\begin{aligned}V_9 =& (2+a_1K)[2(a_0+a_1+a_0a_1K)]^2 - \tfrac{1}{K}a_1\Big[\left(2a_0K+a_0a_1K^2\right) \\ &+\sqrt{\left(2a_0K+a_0a_1K^2\right)^2+4\left(2K+a_1K^2\right)(a_0+a_1+a_0a_1K)}\Big]^2,\end{aligned} \tag{319}$$

and

$$\begin{aligned}W_9 =& \Big[\left(2a_0K+a_0a_1K^2\right)+ \\ &\sqrt{\left(2a_0K+a_0a_1K^2\right)^2+4\left(2K+a_1K^2\right)(a_0+a_1+a_0a_1K)}\Big]^2.\end{aligned} \tag{320}$$

For any given values of a_0, a_1 and K, one has $W_9 > 0$. Therefore, Eq. (318) implies

$$V_9 \leq 0. \tag{321}$$

Hence

$$\begin{aligned}V_9 =& (2+a_1K)[2(a_0+a_1+a_0a_1K)]^2 - \tfrac{1}{K}a_1\Big[\left(2a_0K+a_0a_1K^2\right) \\ &+\sqrt{\left(2a_0K+a_0a_1K^2\right)^2+4\left(2K+a_1K^2\right)(a_0+a_1+a_0a_1K)}\Big]^2 \\ =& 2a_0(2+a_1K)\Big[2a_0+a_1K\left(2\tfrac{1}{K}+2a_0+2a_1+a_0a_1K\right) \\ &-a_1\sqrt{K(2+a_1K)\left\{a_0^2K(2+a_1K)+4[a_1+a_0(1+a_1K)]\right\}}\Big] \leq 0.\end{aligned} \tag{322}$$

Since $2a_0(2+a_1K) > 0$, then, from Eq. (322) the following condition must be met:

$$\begin{aligned}&2a_0+a_1K\left(2\tfrac{1}{K}+2a_0+2a_1+a_0a_1K\right) \\ &-a_1\sqrt{K(2+a_1K)\left\{a_0^2K(2+a_1K)+4[a_1+a_0(1+a_1K)]\right\}} \leq 0.\end{aligned} \tag{323}$$

Therefore,

$$\begin{aligned}&2a_0+a_1K\left(2\tfrac{1}{K}+2a_0+2a_1+a_0a_1K\right) \\ &\leq a_1\sqrt{K(2+a_1K)\left\{a_0^2K(2+a_1K)+4[a_1+a_0(1+a_1K)]\right\}}.\end{aligned} \tag{324}$$

Since both sides of inequality Eq. (324) are positive, then

$$\begin{aligned}&\left[2a_0+a_1K\left(2\tfrac{1}{K}+2a_0+2a_1+a_0a_1K\right)\right]^2 \\ &\leq a_1^2\left[K(2+a_1K)\left\{a_0^2K(2+a_1K)+4[a_1+a_0(1+a_1K)]\right\}\right],\end{aligned} \tag{325}$$

which leads to

$$\begin{aligned}0 \leq& a_1^2\left[K(2+a_1K)\left\{a_0^2K(2+a_1K)+4[a_1+a_0(1+a_1K)]\right\}\right] \\ &-\left[2a_0+a_1K\left(2\tfrac{1}{K}+2a_0+2a_1+a_0a_1K\right)\right]^2 \\ =& -4(a_0+a_1+a_0a_1K)^2.\end{aligned} \tag{326}$$

Thus,

$$-4\,(a_0+a_1+a_0a_1K)^2 \geq 0,$$

which cannot happen due to $a_0+a_1+a_0a_1K > 0$.

Therefore, the assumption was false, so $U_8 > 0$. Thus,

$$W_7 = \frac{U_8}{a_0} > 0, \tag{327}$$

then,

$$V_5 = V_7 + a_0W_7 > 0, \tag{328}$$

which proves that

$$\begin{aligned}\mathscr{P}_1 &> [V_5G1 + 2W_5(b_1-b_0)]\,G1 \\ &\geq V_5G1^2 > 0.\end{aligned} \tag{329}$$

So we have that $\mathscr{P}_1 > 0$, $\mathscr{Q}_1 > 0$ and $\mathscr{R}_1 > 0$, then,

$$V_2 = \mathscr{P}_1 + \mathscr{Q}_1 + \mathscr{R}_1 > 0, \tag{330}$$

and therefore,

$$\lim_{\beta\to 0}(\pi_1^c - \pi_1^*)(\beta) = \frac{1}{2}\frac{V_2}{W_2} > 0, \tag{331}$$

which proves Eq. (222). The proof of the theorem is complete.