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Paper:

Fuzzy Inference Based on α-Cuts and Generalized Mean: Relations Between the Methods in its Family and their Unified Platform

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This paper clarifies the relations in properties and structures between fuzzy inference methods based on α-cuts and the generalized mean. The group of the inference methods is named the α-GEM (α-cut and generalized-mean-based inference) family. A unified platform is proposed for the inference methods in the α-GEM family by the effective use of the above-mentioned relations. For the unified platform, a criterion is made clear to uniquely determine the value of a parameter in fuzzy-constraint propagation control for facts given by singletons. Moreover, conditions are derived to make the inference methods in the α-GEM family equivalent to singleton-consequent-type fuzzy inference which has been successfully applied to a wide variety of fields. Thereby, the unified platform can contribute to the construction of an inference engine for both the methods in the α-GEM family and singleton-consequent-type fuzzy inference. Such scheme of the inference engine provides an effective way to make these inference methods transformed into each other in learning for selecting the inference methods as well as for optimizing fuzzy rules.

Keywords: fuzzy inference, fuzzy rule interpolation, convex fuzzy set, α-cut, generalized mean

1. Introduction

Parallel fuzzy inference has shown its effectiveness in various applications. It performs a nonlinear mapping between fuzzy sets in accordance with a number of fuzzy rules with high understandability. Fuzzy sets represent fuzzy constraints on variables that can be characterized by fuzziness [1, 2] and specificity [3, 4]. The rules for the inference process can be seen as the rules for governing fuzzy-constraint propagation from given facts to consequences [5, 6].

Conventional fuzzy inference has been based on the compositional rule of inference (CRI). In this paper, the inference on the basis of CRI is called *compositional-rule-based inference* (CR-based inference). In CR-based inference, the fuzziness and specificity of deduced consequences tend to be larger and smaller, respectively, in comparison with those of consequent fuzzy sets in activated fuzzy rules. Although this tendency can be viewed as an approved property as in human inference, CR-based inference often deduces consequences with excessively large fuzziness and excessively small specificity. Moreover, it may deduce consequences in nonconvex forms even if the consequent fuzzy sets are all defined by convex fuzzy sets. Such deduced consequences may need to be translated into convex fuzzy sets when they are required to be treated as fuzzy numerical values. Convex fuzzy sets have often been used for representing fuzzy numerical information. The importance of convexity in deduced consequences is more precisely discussed in [7–11]. These properties of CR-based inference stem from the consequence aggregation by disjunctive or conjunctive operators [12, 13].

In order to solve the above-mentioned problems in CR-based inference, a fuzzy inference method has been proposed on the basis of α-cuts (also called α-level sets) and the generalized mean, which is named α-GEM (α-level-set and generalized-mean-based inference) [7]. This method mathematically proves the convexity of consequences. Moreover, it can control the fuzziness and specificity of consequences. The scheme of α-GEM leads to an inference method named α-GEMII (α-level-set and generalized-mean-based inference with the proof of two-sided symmetry of consequences) [8–11, 14]. α-GEMII mathematically proves the symmetricity of consequences under some conditions that are axiomatically derived from the viewpoint of fuzzy inference. Since it has been proposed, α-GEMII has played a central role as the basis of other inference methods [15–21]. Let the group of these inference methods be called *the α-GEM family* [22]. The fuzzy inference methods in the α-GEM family prove the convexity of consequences while suppressing the excessively large fuzziness and excessively small specificity of the consequences.

This paper first clarifies the relations in properties and structures between the inference methods in the α-GEM family [22]. Then, a unified platform is proposed, especially for the most usable methods in the family at present. It is derived by the effective use of the above-mentioned relations. For the unified platform, a criterion in fuzzy-constraint propagation control is made clear to uniquely determine the value of a parameter for facts given by singletons. Moreover, conditions are derived to prove the equivalence between the methods in the α-GEM family and singleton-consequent-type fuzzy inference which has been practically applied to a wide variety of fields. They provide an effective way to construct an inference engine based on the unified platform for both the methods in the α-GEM family and singleton-consequent-type fuzzy inference. Thereby, these inference methods can be transformed into each other on a single base, especially in learning for selecting the inference methods as well as for optimizing fuzzy rules.

The rest of the paper is organized as follows: Section 2 presents the definitions and preliminaries for the following discussions. Section 3 summarizes the common properties of the inference methods in the α-GEM family. Section 4 explains the inference methods for non-sparse rule bases in the α-GEM family which includes α-GEM, α-GEMII, and other related methods originated by α-GEMII. Section 5 explains the inference methods for sparse rule bases in the α-GEM family which are originated by α-GEMII. In these two sections, the relations in properties and structures between the inference methods are clarified. Section 6 proposes a unified platform for the methods in the α-GEM family by the effective use of the relations described in Sections 4 and 5. Section 7 derives the conditions to make the methods for non-sparse rule bases in the α-GEM family equivalent to singleton-consequent-type fuzzy inference which has been successfully applied to many fields. Section 8 derives the conditions to make the methods for sparse rule bases in the α-GEM family equivalent to singleton-consequent-type fuzzy inference. These conditions provide an effective way to make the methods in the α-GEM family and singleton-consequent-type fuzzy inference transformed into each other in the unified platform proposed in Section 6. Section 9 concludes this paper.

2. Definitions and Preliminaries

For the following discussions, some definitions and preliminaries are presented. Further details of each are described in [8, 16, 17].

Definition 1 A fuzzy set A in the space X of real numbers is called *convex* if and only if its membership function $\mu_A(x)$ satisfies

$$\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \mu_A(x_1) \wedge \mu_A(x_2), \quad . \quad . \quad (1)$$
$$0 \leq \lambda \leq 1, \quad x_1 \in X, \quad x_2 \in X,$$

where $\wedge$ denotes the minimum operation. ■

Corollary 1 A fuzzy set is convex if and only if all of its α-cuts are convex. ■

Definition 2 When a convex fuzzy set A is defined by a continuous membership function $\mu_A(x)$ and its α-cuts are all bounded, *the reference point* x_A° of A is defined by using its α-cut A_α as follows:

$$x_A^\circ = \frac{x_\alpha^\ell + x_\alpha^u}{2}, \quad \alpha = \max_x \mu_A(x), \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where x_α^u and x_α^ℓ denote the least upper and the greatest lower bounds of A_α, respectively. ■

In this paper, convex fuzzy sets are all defined by unimodal membership functions defined below:

Definition 3 The membership function $\mu_A(x)$ of a fuzzy set A with its bounded support set $\underline{A}$ is called *unimodal* if and only if it is continuous and satisfies the following conditions [16]:

$$\mu_A(x_1) < \mu_A(x_2), \ x_1 < x_2 \leq x_A^\circ, \ x_1 \in \underline{A}, \ x_2 \in \underline{A}, \quad (3)$$
$$\mu_A(x_1) > \mu_A(x_2), \ x_A^\circ \leq x_1 < x_2, \ x_1 \in \underline{A}, \ x_2 \in \underline{A}, \quad (4)$$

where x_A° denotes the reference point of A. ■

For example, triangular membership functions are unimodal. When a fuzzy set is defined by a unimodal membership function, its α-cuts are closed intervals as can be found from Corollary 1 and Definition 3.

Definition 4 A convex fuzzy set A is *symmetric* if and only if the least upper and the greatest lower bounds of each of its α-cuts are placed symmetrically with respect to the reference point of A for all values of α. A convex fuzzy set is called *asymmetric* if and only if it is not symmetric. ■

Definition 5 When a fuzzy set A, in the universe of discourse X given by a closed interval $[x^\ell, x^u]$ of real numbers, is normal and convex, *the fuzziness* $H_{\rm f}(A)$ of A is defined by the family of its α_i-cuts, where the levels of α, $\alpha_i, (i = 1, 2, \ldots, m)$ are arranged at equal intervals in $[0, 1]$:

$$H_{\rm f}(A) = \frac{2}{m\mathscr{R}(X)} \sum_{i=1}^{m} | \, w_{\alpha_i} - w_{\alpha^\circ} \, |, \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$A_{\alpha_i} = [x_{\alpha_i}^\ell, x_{\alpha_i}^u], \quad x_{\alpha_i}^\ell \in X, \quad x_{\alpha_i}^u \in X,$$
$$A_{\alpha^\circ} = [x_{\alpha^\circ}^\ell, x_{\alpha^\circ}^u], \quad x_{\alpha^\circ}^\ell \in X, \quad x_{\alpha^\circ}^u \in X,$$
$$w_{\alpha_i} = x_{\alpha_i}^u - x_{\alpha_i}^\ell, \quad w_{\alpha^\circ} = x_{\alpha^\circ}^u - x_{\alpha^\circ}^\ell, \quad \mathscr{R}(X) = x^u - x^\ell.$$

Here, A_{α_i} represents the α_i-cuts of A. The symbol A_{α° denotes the α-cut of A at the level $\alpha = \alpha^\circ = 0.5$. If A_{α° is not equal to any of $A_{\alpha_i}, (i = 1, 2, \ldots, m)$, it is generated by the following equation:

$$A_{\alpha^\circ} = \left[\frac{x_{\alpha_i}^\ell + x_{\alpha_{i+1}}^\ell}{2}, \ \frac{x_{\alpha_i}^u + x_{\alpha_{i+1}}^u}{2} \right], \quad \alpha_i < 0.5 < \alpha_{i+1}, \ (6)$$

where $\alpha_i < \alpha_{i+1}, (i = 1, 2, \ldots, m-1)$. ■

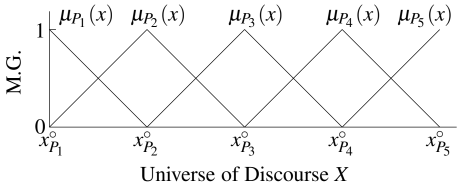

Fig. 1. An example of a TSFP.

Definition 6 When a fuzzy set A, in the universe of discourse X given by a closed interval $[x^{\ell}, x^{u}]$ of real numbers, is normal and convex, *the non-specificity* $H_{\mathrm{ns}}(A)$ of A is defined by the family of its α_i-cuts, where the levels of α, $\alpha_i, (i = 1, 2, \ldots, m)$ are arranged at equal intervals in $[0, 1]$:

$$H_{\mathrm{ns}}(A) = \frac{1}{m\mathscr{R}(X)} \sum_{i=1}^{m} (x^{u}_{\alpha_i} - x^{\ell}_{\alpha_i}), \quad \ldots \ldots \quad (7)$$

where an α_i-cut A_{α_i} of A is given by $A_{\alpha_i} = [x^{\ell}_{\alpha_i}, x^{u}_{\alpha_i}]$. *The specificity* $H_{\mathrm{s}}(A)$ of A is defined as follows:

$$H_{\mathrm{s}}(A) = 1 - H_{\mathrm{ns}}(A). \quad \ldots \ldots \ldots \ldots \quad (8)$$

■

Definition 7 The generalized mean $M(\{x_j, p_j\}; \omega)$ is defined by

$$M(\{x_j, p_j\}; \omega) = \left[\frac{\sum_{j=1}^{n} p_j x_j^{\omega}}{\sum_{j=1}^{n} p_j} \right]^{\frac{1}{\omega}}, \quad x_j > 0, \quad p_j > 0, \quad (9)$$

where x_j denotes a real number in the universe of discourse and p_j represents a real number used for the weight of x_j. The symbol ω denotes a real number to determine the property of the mean [8]. ■

Definition 8 When the universe of discourse X is partitioned with fuzzy sets $P_j, (j = 1, 2, \ldots, n)$ so as to satisfy

$$\sum_{j=1}^{n} \mu_{P_j}(x) = 1, \ \forall x \in X, \quad \ldots \ldots \ldots \quad (10)$$

$$\mu_{P_j}(x) = 1, \ \forall j, \ \exists x \in X, \quad \ldots \ldots \quad (11)$$

such partition is called *a strong fuzzy partition* (SFP). Here, $\mu_{P_j}(x)$ denotes the membership function of P_j. In particular, it is called *a triangular strong fuzzy partition* (TSFP) when $\mu_{P_j}(x)$, $(j = 1, 2, \ldots, n)$ are all triangular membership functions. ■

Fig. 1 shows an example of a TSFP, where the universe of discourse X is partitioned with triangular membership functions $\mu_{P_j}(x), (j = 1, 2, \ldots, 5)$. In this figure, M.G. stands for membership grade. As can be found in **Fig. 1**, a TSFP can be parameterized by using the core sets $x^{\circ}_{P_j}$ of $P_j, (j = 1, 2, \ldots, 5)$.

3. Common Properties of the Inference Methods in the α-GEM Family

The inference methods in the α-GEM family treat the following form of inference:

$$\begin{array}{rl} \text{Fuzzy Rule 1} : & \text{If } x \text{ is } P_1 \text{ then } y \text{ is } Q_1. \\ \text{Fuzzy Rule 2} : & \text{If } x \text{ is } P_2 \text{ then } y \text{ is } Q_2. \\ \vdots & \qquad \vdots \\ \text{Fuzzy Rule } n : & \text{If } x \text{ is } P_n \text{ then } y \text{ is } Q_n. \\ \text{Given Fact} : & x \text{ is } \tilde{P}. \\ \hline \text{Consequence} : & y \text{ is } \tilde{Q}. \end{array}$$

Here, P_j and $\tilde{P}$ denote fuzzy sets in the universe of discourse X, whereas Q_j and $\tilde{Q}$ represent fuzzy sets in the universe of discourse Y. In particular, P_j in the antecedent part of the fuzzy rule is called *an antecedent fuzzy set*, whereas Q_j in the consequent part of the fuzzy rule is called *a consequent fuzzy set*. When Q_j is defined by a singleton, it is especially called *a consequent singleton* in order to emphasize that it is not a fuzzy set in the meaning of a non-singleton although the definition of fuzzy sets mathematically includes that of singletons. When P_j is defined by a singleton, particularly for sparse rule bases, it is especially called *an antecedent singleton* for the same reason. The membership functions of P_j, Q_j, $\tilde{P}$, and $\tilde{Q}$ are denoted by $\mu_{P_j}(x)$, $\mu_{Q_j}(y)$, $\mu_{\tilde{P}}(x)$, and $\mu_{\tilde{Q}}(y)$, respectively. In this paper, P_j, Q_j, and $\tilde{P}$ are all defined by normal and convex fuzzy sets whose membership functions are unimodal [16] and their reference points are placed in $[0, 1]$. For convenience, the fuzzy rule "If x is P_j then y is Q_j" is denoted by "$P_j \Rightarrow Q_j$."

The inference methods in the α-GEM family have the following common properties [22]:

(i) They mathematically prove the convexity of deduced consequences. Thereby, the deduced consequences can be treated as fuzzy numerical values.

(ii) They suppress the excessively large fuzziness and excessively small specificity of deduced consequences by using the fuzzy-constraint propagation control. Thereby, they can provide a more precise nonlinear-mapping between convex fuzzy sets in accordance with the distribution forms of fuzzy rules.

(iii) Their operations can be performed independently at each level of α. The independency contributes to their fast computations with parallel processing for operations at each level of α.

Properties (i) and (ii) are especially effective in comparison with CR-based inference as stated in Section 1.

As shown in Property (iii), the operations for the inference methods in the α-GEM family are α-cut based. For the following discussions, some symbols are defined in relation to α-cuts: The α-cuts of P_j, Q_j, $\tilde{P}$, and $\tilde{Q}$ are represented by $P_{j\alpha}$, $Q_{j\alpha}$, $\tilde{P}_{\alpha}$, and $\tilde{Q}_{\alpha}$, respectively. Since

P_j, Q_j, and $\tilde{P}$ are defined by unimodal membership functions, $P_{j\alpha}$, $Q_{j\alpha}$, and $\tilde{P}_\alpha$ are closed intervals, which are denoted by $[x^\ell_{P_{j\alpha}}, x^u_{P_{j\alpha}}]$, $[x^\ell_{Q_{j\alpha}}, x^u_{Q_{j\alpha}}]$, and $[x^\ell_{\tilde{P}_\alpha}, x^u_{\tilde{P}_\alpha}]$, respectively. Because the inference methods in the α-GEM family prove the convexity of $\tilde{Q}$, $\tilde{Q}_\alpha$ is also a closed interval. Let the closed interval be denoted by $[x^\ell_{\tilde{Q}_\alpha}, x^u_{\tilde{Q}_\alpha}]$.

4. Inference Methods for Non-Sparse Rule Bases in the α-GEM Family

This section describes the inference methods for non-sparse rule bases in the α-GEM family together with a related method [22]. The relations in properties and structures between these inference methods are clarified.

4.1. Fuzzy Inference Based on the Weighted Arithmetic Mean of Fuzzy Sets: α-RITHM — The Origination of α-GEM —

An inference method on the basis of the arithmetic mean of fuzzy sets has been proposed in [23]. It deduces consequences with the arithmetic mean of consequent fuzzy sets weighted by the compatibility degrees. The mean operations can be performed via α-cuts on the basis of the extension principle. In this paper, the inference method is named *α-RITHM* (α-level-set and arithmetic-mean-based inference). α-RITHM leads to α-GEM which includes α-RITHM as a special case as shown in Section 4.2.

α-RITHM mathematically proves the convexity of deduced consequences when the consequent fuzzy sets are all defined by convex fuzzy sets. Therefore, the α-cuts of the consequences are given by closed intervals. α-RITHM deduces consequences by using the following equations:

$$y^u_{\tilde{Q}_\alpha} = \frac{\sum_{j=1}^{n} \tilde{p}_j \, y^u_{Q_{j\alpha}}}{\sum_{j=1}^{n} \tilde{p}_j}, \quad (12)$$

$$y^\ell_{\tilde{Q}_\alpha} = \frac{\sum_{j=1}^{n} \tilde{p}_j \, y^\ell_{Q_{j\alpha}}}{\sum_{j=1}^{n} \tilde{p}_j}, \quad (13)$$

where $\tilde{p}_j$ denotes the compatibility degree between P_j and $\tilde{P}$. In particular, the compatibility degree $\tilde{p}_j$ can be defined by

$$\tilde{p}_j = \sup_x [\mu_{\tilde{P}}(x) \wedge \mu_{P_j}(x)]. \quad (14)$$

This definition is adopted in this paper because Eq. (14) can be calculated via α-cuts [24]. The operations for α-RITHM are α-cut based as shown in Eqs. (12) and (13). They can be performed independently at each level of α. This property contributes to their fast computations with parallel processing for operations at each level of α. If the need arises, $\tilde{Q}$ can be obtained from $\tilde{Q}_\alpha$ on the basis of the resolution identity theorem.

When $Q_j, (j = 1, 2, \ldots, n)$ are given by singletons with real numbers $y_j, (j = 1, 2, \ldots, n)$, Eqs. (12) and (13) turn to

$$y_{\tilde{Q}} = y^u_{\tilde{Q}_\alpha} = y^\ell_{\tilde{Q}_\alpha} = \frac{\sum_{j=1}^{n} \tilde{p}_j \, y_j}{\sum_{j=1}^{n} \tilde{p}_j}, \quad \forall \alpha \in (0,1]. \quad (15)$$

This is because $y^u_{Q_{j\alpha}} = y^\ell_{Q_{j\alpha}} = y_j$, $\forall \alpha \in (0,1]$. Eq. (15) means that α-RITHM deduces a singleton with a real number $y_{\tilde{Q}}$. The inference method using Eq. (15) has been applied to a wide variety of fields that require singleton consequences. For convenience, let this inference method be called *singleton-consequent-type fuzzy inference* in this paper. From the discussion above, it can be found that α-RITHM includes singleton-consequent-type fuzzy inference as a special case.

4.2. α-Cut and Generalized-Mean-Based Inference: α-GEM

In α-RITHM, the fuzzy constraints of deduced consequences are determined only by those of consequent fuzzy sets and not by the distribution forms of consequent fuzzy sets. Even if the fuzziness and specificity of given facts are varied, those of deduced consequences are not changed accordingly. For example, even when the specificity of given facts is made smaller, that of the deduced consequences does not become smaller according to the distribution forms of the consequent fuzzy sets in activated fuzzy rules. That is, the fuzzy constraints of given facts are not propagated to those of consequences. In order to make the fuzzy-constraint propagation controllable in fuzzy inference, an inference method has been proposed in [7], which is based on α-cuts and the generalized mean. It is called α-GEM (α-level-set and generalized-mean-based inference). α-GEM can control the fuzzy-constraint propagation from given facts to consequences, while proving the convexity of the consequences.

α-GEM deduces consequences by using the following equations:

$$y^u_{\tilde{Q}_\alpha} = M(\{y^u_{Q_{j\alpha}}, \tilde{p}_j\}; \omega(\alpha)), \quad (16)$$

$$y^\ell_{\tilde{Q}_\alpha} = \bar{M}(\{y^\ell_{Q_{j\alpha}}, \tilde{p}_j\}; \omega(\alpha)). \quad (17)$$

Here, $\omega(\alpha)$ is a non-increasing function of α and $\omega(\alpha) \geq 1$. The function $\bar{M}(\{x_j, p_j\}; \omega)$ is defined by

$$\bar{M}(\{x_j, p_j\}; \omega) = 1 - M(\{1 - x_j, \, p_j\}; \omega). \quad (18)$$

The operation with $\bar{M}$ is called *the dual operation of M* in this paper. In Eqs. (16) and (17), $\omega(\alpha)$ determines the degree to which the fuzzy constraints of given facts are propagated to those of consequences. If the need arises, $\tilde{Q}$ can be obtained from $\tilde{Q}_\alpha$ on the basis of the resolution identity theorem.

α-GEM turns to α-RITHM when $\omega(\alpha) = 1$, $\forall \alpha \in (0,1]$. That is, α-GEM includes α-RITHM as a special

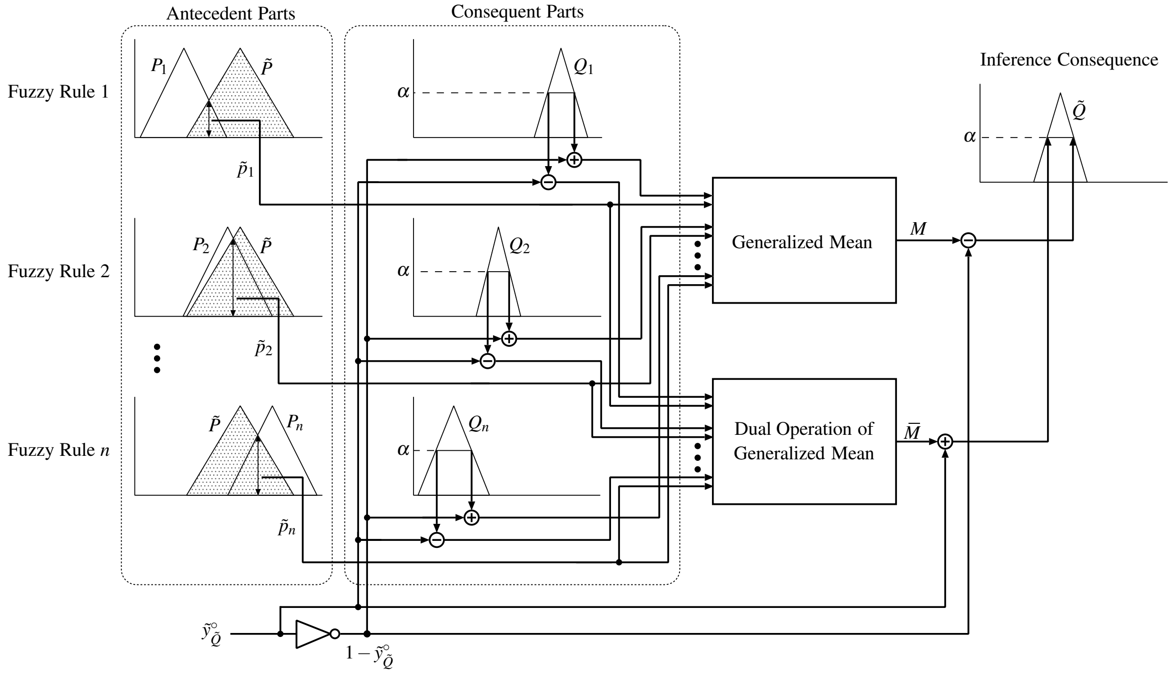

Fig. 2. The computational process of α-GEMII [8].

case. Since α-RITHM includes singleton-consequent-type fuzzy inference as a special case, α-GEM includes both α-RITHM and singleton-consequent-type fuzzy inference as special cases. More precise discussions on the fuzzy-constraint propagation in α-GEM are presented in [7].

4.3. Inference with the Proof of Symmetricity of Consequences: α-GEMII

4.3.1. α-GEMII and its Effectiveness

Although α-GEM can control the fuzzy-constraint propagation while proving the convexity of deduced consequences, it does not prove the symmetricity of the deduced consequences even under some conditions that are axiomatically derived from the viewpoint of fuzzy inference. In order to solve this problem, an inference method has been proposed in [8, 9], which is called α-GEMII (α-level-set and generalized-mean-based inference with the proof of two-sided symmetry of consequences).

The operational steps of α-GEMII are shown below [8]:

Step 1a: Obtain the compatibility degree $\tilde{p}_j$ between $\tilde{P}$ and P_j.

Step 2a: Calculate the weighted arithmetic mean $y^{\circ}_{\tilde{Q}}$ of the reference points $y^{\circ}_{Q_j}$ of Q_j as follows:

$$y^{\circ}_{\tilde{Q}} = M(\{y^{\circ}_{Q_j}, \tilde{p}_j\}; 1). \quad (19)$$

Step 3a: Obtain the closed interval $[y^{\ell}_{\tilde{Q}_\alpha}, y^{u}_{\tilde{Q}_\alpha}]$ as the α-cut $\tilde{Q}_\alpha$ of the consequence $\tilde{Q}$, by using the following equations:

$$y^{u}_{\tilde{Q}_\alpha} = M(\{y^{u}_{Q_{j\alpha}} + (1 - y^{\circ}_{\tilde{Q}}), \tilde{p}_j\}; \omega(\alpha)) - (1 - y^{\circ}_{\tilde{Q}}), \quad (20)$$

$$y^{\ell}_{\tilde{Q}_\alpha} = \overline{M}(\{y^{\ell}_{Q_{j\alpha}} - y^{\circ}_{\tilde{Q}}, \tilde{p}_j\}; \omega(\alpha)) + y^{\circ}_{\tilde{Q}}, \quad (21)$$

where $\omega(\alpha)$ denotes a non-increasing function of α and $\omega(\alpha) \geq 1$. The automatic generation of $\omega(\alpha)$ is described in Section 4.3.2.

Step 4a: If the need arises, obtain $\tilde{Q}$ from $\tilde{Q}_\alpha$ on the basis of the resolution identity theorem.

Fig. 2 shows the computational process of α-GEMII. When $\omega(\alpha) = 1$, $\forall\alpha \in (0, 1]$, α-GEMII turns to α-RITHM. Since α-RITHM includes singleton-consequent-type fuzzy inference as discussed in Section 4.1, α-GEMII includes both α-RITHM and singleton-consequent-type fuzzy inference as special cases.

The symmetricity of deduced consequences holds as shown in the following proposition which is mathematically proved in [8]:

Proposition 1 α-GEMII, whose operations are shown in Steps 1a–4a, deduces consequences in symmetric forms when the following conditions are satisfied:

(i) Consequent fuzzy sets are all normal, convex, and symmetric.

(ii) The membership-function shapes of consequent fuzzy sets are symmetric with respect to the point $\tilde{y}^{\circ}_{\tilde{Q}}$ in the universe of discourse Y.

(iii) The compatibility degree $\tilde{p}_{j_1}$ is equal to $\tilde{p}_{j_2}$, where the consequent fuzzy set in the j_1-th fuzzy rule and that in the j_2-th fuzzy rule are placed symmetrically with respect to $\tilde{y}^{\circ}_{\tilde{Q}}$. ■

α-GEMII has sensitivity to the symmetricity of the distribution forms of consequent fuzzy sets and compatibility degrees. Conditions (i)–(iii) make the distribution forms and compatibility degrees symmetric with respect to $\tilde{y}^{\circ}_{\tilde{Q}}$. If Conditions (i)–(iii) are not satisfied and nevertheless consequences are always deduced with symmetric fuzzy sets, it means that such inference operations are insensitive to the distribution forms of both the consequent fuzzy sets and the compatibility degrees. Therefore, Conditions (i)–(iii) have their validity. α-GEMII is designed so as to reflect the distribution forms of both the consequent fuzzy sets and the compatibility degrees to the forms of the deduced consequences. The symmetricity of the deduced consequences is more precisely discussed in [8]. Moreover, comparative analyses between α-GEMII and α-GEM are presented in [8].

α-GEMII can be performed for deducing the support sets of consequences in exactly the same way as for the α-cuts of consequences. This is because the support set $\underline{A}$ of a fuzzy set A can be regarded as an α-cut of A [8].

4.3.2. Control of Fuzzy-Constraint Propagation

The non-increasing function $\omega(\alpha)$ controls the fuzzy-constraint propagation from given facts to consequences. Although α is greater than 0, the value of $\omega(0)$ is interpreted as that of the parameter ω in the generalized mean for deducing the support set $\underline{\tilde{Q}}$ of $\tilde{Q}$ in Step 3a. The non-increasing function $\omega(\alpha)$ is automatically generated by using the scheme shown below [9]:

Step 1b: Compose the fuzzy rules "If x is P_j then x is P_j." $(j = 1, 2, \dots, n)$. Let the composed fuzzy rules be called *fuzzy tautological rules* (FTRs) [9].

Step 2b: Perform α-GEMII, following Steps 1a–4a with a given fact $\tilde{P}$ and the FTRs. Let the deduced consequence be denoted by $\tilde{P}'$. Tune $\omega(\alpha)$ so as to minimize the difference in fuzziness and specificity between $\tilde{P}$ and $\tilde{P}'$ by using the α-cuts of $\tilde{P}$ and $\tilde{P}'$, under the following conditions:

$$\omega(\alpha') \geq \omega(\alpha''), \quad \alpha' < \alpha'', \quad \dots\dots (22)$$

$$\omega(\alpha) \geq 1. \quad \dots\dots (23)$$

Step 3b: Deduce $\tilde{Q}_\alpha$, following Steps 1a–4a with the tuned $\omega(\alpha)$.

The minimization in Step 2b can be formalized in many different ways. This scheme provides flexibility in designing the inference based on α-GEMII for given systems. In [9], the genetic algorithms are applied to the minimization and the values of the related parameters are presented.

By using the propagation control with $\omega(\alpha)$, α-GEMII can deduce convex fuzzy sets, while the fuzzy constraints of given facts are propagated to those of the deduced consequences. Thereby, α-GEMII performs a nonlinear mapping between convex fuzzy sets. The membership-function shapes and positions of the given facts are nonlinearly transformed in accordance with fuzzy rules for deducing the consequences. In addition, α-GEMII can avoid the excessive fuzziness increase and specificity decrease in deducing the consequences [8, 9]. The other properties of α-GEMII are discussed in [10, 11, 14]. Comparative studies between CR-based inference and α-GEMII with the fuzzy-constraint propagation control are presented in [9].

4.3.3. Formulation of α-GEMII for Triangular Membership Functions

Triangular membership functions are practically important because of their high computational efficiency in inference operations. The triangular membership function of a fuzzy set A is parameterized by using its two α-cuts, namely the α-cut at $\alpha = 1$ and the support set. The least upper and the greatest lower bounds of these two α-cuts form the triangular membership function as a piecewise linear function [15]. Accordingly, α-GEMII can be performed with triangular membership functions by the effective use of α-cut-based operations.

It should be noted that the α-cut at $\alpha = 1$ is a singleton when a normal fuzzy set is defined by a triangular membership function. This implies that the least upper bound of the α-cut is equal to its greatest lower bound at $\alpha = 1$. α-GEMII is proved to deduce the consequence $\tilde{Q}$ with a triangular membership function when $\omega(1) = 1$ [8, 15]. Thus, the value of $\omega(1)$ is fixed to 1, whereas the value of $\omega(0)$ is to be tuned to control the fuzzy-constraint propagation in Steps 1b and 2b. For the following discussions, let ω_0 denote the tuned value of $\omega(0)$.

4.4. Synergy with CRI in Fuzzy-Constraint Propagation: α-GEMS

α-GEMII has limitations in the control of fuzzy-constraint propagation although it can perform a nonlinear mapping in accordance with a number of fuzzy rules. The limitations stem from Relations (22) and (23) which are required to be satisfied for proving the convexity of deduced consequences. For the following discussions, let $\tilde{P}_\circ$ denote the consequence deduced with α-GEMII and FTRs at $\omega(\alpha) = 1$ for all levels of α. Even when the support-set specificity of $\tilde{P}$ is varied in the range larger than that of $\tilde{P}_\circ$, the support-set specificity of $\tilde{Q}$ is not changed accordingly. The limitations in the fuzzy-constraint propagation lead to the difficulty in constructing the models of various given systems. In contrast, CR-based inference can rather easily provide various fuzzy-constraint propagation with high understandability, particularly when a single fuzzy rule is applied. It is difficult, however, to perform a nonlinear mapping between convex fuzzy sets in accordance with a number of fuzzy rules in parallel. In order to solve the problems in α-GEMII and CR-based inference at the same time, an inference method has been proposed in [15], which is called α-GEMS (α-cut and generalized-mean-based inference in synergy with composition).

α-GEMS can propagate the fuzzy constraints of given fact to those of deduced consequences in various ways. In α-GEMS, a reference fuzzy rule is generated for each given fact via a nonlinear mapping between convex fuzzy sets by using α-GEMII. Thereby, the fuzzy-constraint propagation can be conducted while suppressing the excessively large fuzziness and excessively small specificity of consequences. For further control of the fuzzy-constraint propagation, the CRI-based operations are performed with the reference fuzzy rule and the given fact. This scheme makes it possible to provide various fuzzy-constraint propagation by selecting one pair from the widely available implications and compositional operations. The CRI-based operations are also α-cut based and can be conducted independently at each level of α, particularly when the max-min composition is adopted [24]. In [15], α-GEMS is compared with α-GEMII in simulations and it shows the feasibility to provide various fuzzy-constraint propagation. Although α-GEMS is computationally simple, it can be performed only with symmetric fuzzy sets.

4.5. Synergy with CRI via Linguistic-Truth-Value Control: α-GEMST

As stated in Section 4.4, α-GEMS cannot be conducted with asymmetric fuzzy sets; it can be performed only with symmetric fuzzy sets. In order to cope with asymmetric fuzzy sets as well as with symmetric ones, an inference method has been proposed in [16], which is called α-GEMST (α-level-set and generalized-mean-based inference in synergy with composition via linguistic-truth-value control).

α-GEMST suppresses excessive specificity decreases in consequences, particularly stemming from the asymmetricity in generating a reference fuzzy rule for each given fact. The control process for the suppression is theoretically derived by evaluating the support-set specificity of consequences. The evaluations are conducted via linguistic truth values (LTVs) which are generated in the course of the CRI-based operations. Thereby, α-GEMST reflects the fuzzy constraints of given facts to those of consequences, to a feasible extent. The fuzzy-constraint propagation is further controlled by using the CRI-based operations with the reference fuzzy rule and the given fact. In this scheme, α-GEMST can also provide various fuzzy-constraint propagation by selecting one pair from the widely available implications and compositional operations. The CRI-based operations can be performed via α-cuts independently at each level of α as mathematically proved in [16].

5. Inference Methods for Sparse Rule Bases in the α-GEM Family

This section explains the inference methods for sparse rule bases in the α-GEM family [22]. The relations in properties and structures are made clear between the inference methods.

5.1. Fuzzy Rule Interpolation at Multi-Level Points: α-GEMMI

Fuzzy rule interpolation was introduced first in [25]. Since then, other interpolation methods have been developed [17–20, 26–30]. The effectiveness of interpolation approaches is also illustrated with some applications [31, 32].

Conventional inference methods for sparse rule bases focus only on interpolation by using fuzzy rules whose antecedent fuzzy sets are adjacent to given facts, while the convexity in deduced consequences is taken into account. In some cases, however, given facts activate a number of fuzzy rules even in sparse rule bases. When such activated fuzzy rules represent nonlinear relations, the membership-function shapes and positions of given facts are to be nonlinearly transformed in accordance with the fuzzy rules for deducing consequences. Nevertheless, the conventional inference methods for sparse rule bases cannot perform the nonlinear transformation with a number of activated fuzzy rules. They may deduce consequences with excessively large fuzziness and excessively small specificity even if the activated fuzzy rules do not represent the possibility of such fuzziness and specificity with the consequent fuzzy sets [17]. In order to solve this problem, an inference method has been proposed in [17], which is called α-GEMMI (α-level-set and generalized-mean-based inference with multi-level interpolation).

In α-GEMMI, fuzzy rules are interpolated at a number of points. These points are called *multi-level points* which are also proposed in [17]. The multi-level points are generated by the minimum number of α-cuts with which each given fact can be defined. They are provided by the least upper and the greatest lower bounds of the α-cuts. The membership function of each given fact is represented by a piecewise linear function which is formed by linearly connecting the least upper and the greatest lower bounds of the α-cuts [18]. It can be seen that the least upper and the greatest lower bounds of the α-cuts characterize the fuzzy constraint of the given fact. Thereby, α-GEMMI reflects the fuzzy constraints of given facts to those of deduced consequences. Fuzzy rule interpolation at the multi-level points is called *multi-level interpolation*. The fuzzy rule interpolation in α-GEMMI is also conducted by using α-GEMII at $\omega(\alpha) = 1$ for all levels of α [17].

In α-GEMMI, α-GEMII is performed with fuzzy rules interpolated at the multi-level points. This process makes it possible to nonlinearly transform the membership-function shapes and positions of given facts for deducing consequences in accordance with sparse fuzzy rules. The properties of α-GEMMI are demonstrated and compared with those of conventional inference methods for sparse rule bases in [17].

5.2. Fuzzy Rule Interpolation with Extended Multi-Level Points: α-GEMMIET

In order to increase the mapping accuracy in sparse rule bases, an inference method has been proposed in [18], which applies the scheme of α-GEMMI. In this paper, it is named *α-GEMMIET* (α-level-set and generalized-mean-based inference with multi-level interpolation extended in the number of points for interpolation).

In α-GEMMIET, the number of α-cuts of each given fact is increased in generating the interpolation points so that it is larger than the minimum number of α-cuts with which each given fact can be defined. The interpolation points are provided by the least upper and the greatest lower bounds of the α-cuts whose number is increased in the above-mentioned way. In this paper, such interpolation points are named *extended multi-level points* in order to differentiate them from the multi-level points stated in Section 5.1. The fuzzy rule interpolation at the extended multi-level points is named *extended multi-level interpolation* in this paper.

In α-GEMMIET, consequences are deduced by using α-GEMII with fuzzy rules interpolated at the extended multi-level points. As the number of α-cuts of each given fact is made larger in generating the extended multi-level points, the number of interpolated fuzzy rules is increased accordingly. Thereby, α-GEMMIET provides a more precise nonlinear-mapping between convex fuzzy sets, reflecting the distribution forms of the sparse fuzzy rules. Simulation results show that the consequence converges as the number of extended multi-level points is made larger by increasing the number of α-cuts of each given fact.

5.3. Fuzzy Rule Interpolation at Infinite-Level Points: α-GEMILIE

In order to further improve the mapping accuracy in accordance with sparse fuzzy rules, an inference method has been proposed in [19]. In this paper, it is named *α-GEMILIE* (α-level-set and generalized-mean-based inference with infinite-level interpolation).

In α-GEMILIE, the interpolation points are provided by the least upper and the greatest lower bounds of the α-cuts of each given fact at an infinite number of levels of α. Such interpolation points are called *infinite-level points*. The fuzzy rule interpolation at the infinite-level points is called *infinite-level interpolation*. α-GEMILIE deduces consequences on the basis of α-GEMII with fuzzy rules interpolated at the infinite-level points.

In α-GEMILIE, convergent consequences are theoretically derived in making the number of extended multi-level points larger by increasing the number of levels of α for the α-cuts of given facts. This increase makes the nonlinear mapping more precise in reflecting the distribution forms of sparse fuzzy rules to consequences. Moreover, the convergent consequences make it unnecessary to examine the number of levels of α for improving the mapping accuracy. The effectiveness and feasibility of α-GEMILIE are detailed in [19].

The authors continue to study inference methods based on infinite-level interpolation although α-GEMILIE has a problem in generating interpolation points as described later in Section 5.4. This is because infinite-level interpolation has feasibility in effectively reflecting the membership-function shapes of given facts to those of deduced consequences.

5.4. Fuzzy Rule Interpolation at an Infinite Number of Activating Points: α-GEMINAS

The extended multi-level points are generated by using the least upper and the greatest lower bounds of the α-cuts of each given fact. The distribution form of these points reflects the membership-function shape and position of the given fact. This property is effective in the nonlinear transformation of the membership-function shapes and positions of the given facts for deducing consequences. The extended multi-level points, however, cannot be generated if there exist intervals in X where the differential values of the membership functions of given facts are zero. This is an obstacle in performing α-GEMMIET and α-GEMILIE in such a case. In order to solve this problem, an inference method has been proposed in [20]. It is called *α-GEMINAS* (α-level-set and generalized-mean-based inference with fuzzy rule interpolation at an infinite number of activating points).

α-GEMINAS interpolates fuzzy rules at an infinite number of *activating points* which were also proposed in [20]. The activating points are generated so as to activate the interpolated fuzzy rules by each given fact. They do not depend on the membership-function shapes of given facts and therefore the above-mentioned problems in α-GEMMIET and α-GEMILIE are solved. In α-GEMINAS, α-GEMII is performed by using all of the interpolated fuzzy rules activated by the given fact. Thereby, the fuzzy constraints of given facts are effectively propagated to those of deduced consequences even with sparse fuzzy rules.

Convergent consequences are theoretically derived in increasing the number of activating points. An infinite number of activating points lead to a more precise nonlinear-mapping between convex fuzzy sets in accordance with sparse fuzzy rules, compared with the case where a finite number of activating points are adopted. They also make it unnecessary to determine the number of activating points in order to satisfy the accuracy required in the nonlinear mapping. Comparative studies between α-GEMINAS and α-GEMILIE are presented in [20].

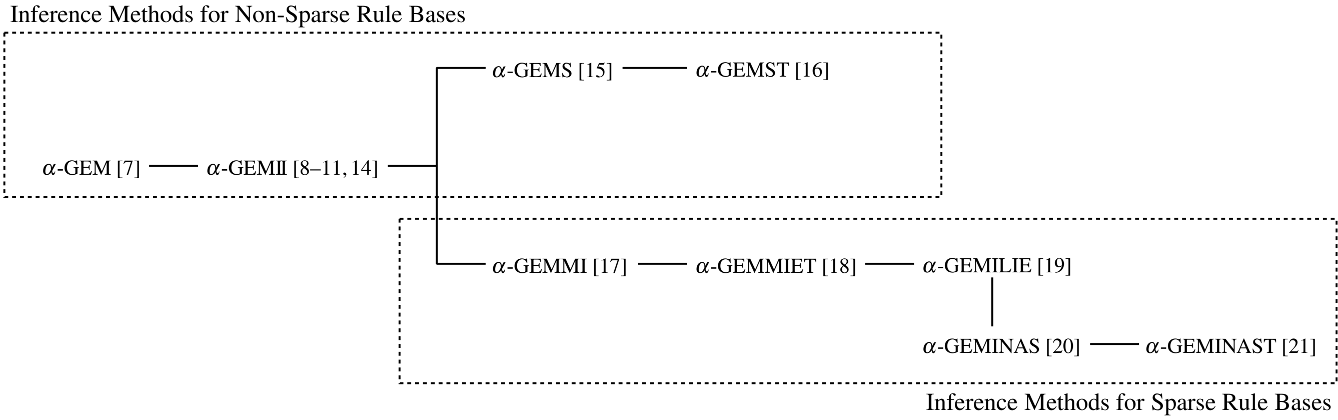

Fig. 3. Relations between the inference methods in the α-GEM family.

5.5. Synergy with CRI via LTV Control in Sparse Rule Bases: α-GEMINAST

α-GEMST can provide various fuzzy-constraint propagation from given facts to consequences with a number of fuzzy rules in parallel, by selecting one pair from the widely available implications and compositional operations. α-GEMST, however, has limitations in controlling fuzzy-constraint propagation from given facts to deduced consequences via the nonlinear mapping represented by fuzzy rules. It cannot always perform the nonlinear mapping, depending on the relation between each given fact and the antecedent fuzzy sets in terms of their support-set specificity. In α-GEMST, the support-set specificity of the antecedent fuzzy sets cannot be arranged independently of the number of fuzzy rules and of the positions of the antecedent fuzzy sets in the input space (referred to as the universe of discourse X in Section 3). This is because the antecedent fuzzy sets are required to cover the entire area of the input space so that any given facts can activate at least one fuzzy rule. This restriction leads to the limitations in arranging the support-set specificity of the antecedent fuzzy sets to provide the nonlinear mapping.

In order to solve the above-mentioned problem in α-GEMST, a fuzzy inference method has been proposed in synergy between α-GEMINAS and the fuzzy-logic-based control for fuzzy-constraint propagation [21]. The inference method is called *α-GEMINAST* (α-level-set and generalized-mean-based inference in synergy between α-GEMINAS and the composition via LTV control). α-GEMINAST can fully control the degree to which the fuzzy constraints of given facts are propagated to those of deduced consequences via the nonlinear mapping represented by fuzzy rules. The propagation degree is determined by the support-set specificity of the antecedent fuzzy sets of the fuzzy rules. As the support-set specificity of the antecedent fuzzy sets is larger, the propagation degree is increased accordingly. α-GEMINAST can arrange the support-set specificity of the antecedent fuzzy sets independently of the number of fuzzy rules and of the positions of the antecedent fuzzy sets in the input space. This is because it can be performed with sparse fuzzy rules as well as with non-sparse fuzzy rules. Thereby, the fuzzy-constraint propagation can be fully controlled to a different degree with each fuzzy rule. It can be interpreted as follows: The support-set specificity of antecedent fuzzy sets can represent how the fuzzy rules themselves are specific, independently of the number of fuzzy rules and of the positions of the antecedent fuzzy sets in the input space. As the support-set specificity of given facts is smaller than that of the antecedent fuzzy sets, the support-set specificity of deduced consequences is decreased accordingly, reflecting the nonlinear mapping in accordance with the fuzzy rules. α-GEMST cannot provide such properties.

After the above-mentioned nonlinear mapping, fuzzy-logic-based control is performed for further fuzzy-constraint propagation with α-cuts in the same way as in α-GEMST. The control method for the fuzzy-constraint propagation is derived on the basis of CRI in which the max-min composition is adopted with either the Gaines-Rescher implication or the Gödel implication. Thereby, α-GEMINAST reflects the fuzzy constraints of given facts to those of consequences to a feasible extent in accordance with a number of fuzzy rules. The effective use of the fuzzy-logic-based scheme also leads to the following: α-GEMINAST can provide various fuzzy-constraint propagation for modeling given systems by selecting one pair from the widely available implications and compositional operations, in performing a nonlinear mapping between convex fuzzy sets even with a number of sparse fuzzy rules.

In α-GEMINAST, the fuzzy-constraint propagation is controlled at the multi-level of α in its α-cut-based operations. The α-cut-based operations can be performed independently at each level of α. The independency leads to the fast computation of α-GEMINAST with parallel processing for each level of α.

Figure 3 shows the relations between the inference methods in the α-GEM family, including α-GEMINAST. The α-GEM family is growing toward other related methods and its applications.

6. A Unified Platform for the Methods in the α-GEM Family

This section proposes a unified platform for the inference methods in the α-GEM family, in particular, α-GEMII, α-GEMST, α-GEMINAS, and α-GEMINAST. These inference methods are most usable in the α-GEM family at present. Although the other methods in the α-GEM family can also be integrated into the unified platform, this paper focuses on these inference methods from a practical viewpoint and thereby simplifies the discussions. The unified platform is derived by the effective use of the relations in properties and structures between the inference methods discussed in Sections 4 and 5. It contributes to the implementation of an inference engine for performing these inference methods in a single base.

In presenting the operational process in the unified platform, the following conditions are imposed for simplicity in practice and basic studies:

(i) Antecedent fuzzy sets P_j and consequent fuzzy sets Q_j, $(j = 1,2,\ldots,n)$ are normal and convex. Each of them is defined by a triangular membership function that is parameterized by the least upper and the greatest lower bounds of its support set and α-cut at $\alpha = 1$. In this case, the α-cut at $\alpha = 1$ is a singleton which is equal to the reference point. When non-sparse rule bases are required, the antecedent fuzzy sets $P_j, (j = 1,2,\ldots,n)$ in each rule base form a TSFP.

(ii) Each given fact $\tilde{P}$ is normal and convex. It is defined by a unimodal membership function that is parameterized with the least upper and the greatest lower bounds of its support set and α-cuts at the multi-level of α. The membership function of each given fact is represented by a piecewise linear function that is formed by linearly connecting the least upper and the greatest lower bounds of the α-cuts [18]. When facts are singletons, their α-cuts are singletons at all levels of α, according to the definition of α-cuts.

These conditions make it possible to control the propagation degree of the fuzzy constraints only on the basis of the support-set widths of P_j and $\tilde{P}$ which determine their support-set specificity. They contribute to the efficient operations for α-GEMII, α-GEMST, α-GEMINAS, and α-GEMINAST.

The operational steps for the unified platform are presented as follows:

Step 1c: Select α-GEMII for deducing consequences by α-GEMII itself or by α-GEMST; select α-GEMINAS for deducing consequences by α-GEMINAS itself or by α-GEMINAST. Note that inference consequences with α-GEMII and α-GEMINAS are also obtained in the course of the operations for α-GEMST and α-GEMINAST in the unified platform, respectively.

Step 2c: Define an implication and a compositional operation for the CRI-based control of the fuzzy-constraint propagation, according to the required inference properties for a given system to be modeled. When the max-min composition is adopted, the operations for CRI can be performed independently at each level of α [24]. The effective use of the independency provides a fast computation with parallel processing for each level of α. The truth function of the selected implication $T_{p\to q}$ is denoted by $p \to q$.

Step 3c: Perform the selected inference method with FTRs and $\tilde{P}$ at $\omega(0) = \omega(1) = 1$. The deduced consequence and its membership function are denoted by $\tilde{P}_\circ$ and $\mu_{\tilde{P}_\circ}(x)$, respectively. In this case, $\mu_{\tilde{P}_\circ}(x)$ is obtained with a triangular membership function because $\mu_{P_j}(x), (j = 1,2,\ldots,n)$ are all triangular. Then, the α-cut of $\tilde{P}_\circ$ at $\alpha = 1$ is a singleton and it leads to the same discussions as in Section 4.3.3.

Step 4c: Calculate the support-set widths $w_{\underline{\tilde{P}}}$ of $\tilde{P}$ and $w_{\underline{\tilde{P}}_\circ}$ of $\tilde{P}_\circ$ by using

$$w_{\underline{\tilde{P}}} = x^u_{\underline{\tilde{P}}} - x^\ell_{\underline{\tilde{P}}},$$
$$w_{\underline{\tilde{P}}_\circ} = x^u_{\underline{\tilde{P}}_\circ} - x^\ell_{\underline{\tilde{P}}_\circ}.$$

Here, $x^u_{\underline{\tilde{P}}}$ and $x^\ell_{\underline{\tilde{P}}}$ respectively denote the least upper and the greatest lower bounds of the support set $\underline{\tilde{P}}$ of $\tilde{P}$, whereas $x^u_{\underline{\tilde{P}}_\circ}$ and $x^\ell_{\underline{\tilde{P}}_\circ}$ respectively represent the least upper and the greatest lower bounds of the support set $\underline{\tilde{P}}_\circ$ of $\tilde{P}_\circ$.

Step 5c: If $w_{\underline{\tilde{P}}} \le w_{\underline{\tilde{P}}_\circ}$, proceed to Step 5c-1 below; otherwise, proceed to Step 6c:

Step 5c-1: If the selected inference method is either α-GEMII or α-GEMINAS for deducing consequences by the selected inference method itself, proceed to Step 5c-3; otherwise, follow Steps 5c-2 and 5c-3:

Step 5c-2: Generate a fuzzy set $P_\circ$ by shifting the position of $\tilde{P}_\circ$ so as to satisfy the following equation [16, 21]:

$$\begin{cases} \mu_{P_\circ}(x^u_{\underline{\tilde{P}}}) = \mu_{P_\circ}(x^\ell_{\underline{\tilde{P}}}), & x^u_{\underline{\tilde{P}}} \ne x^\ell_{\underline{\tilde{P}}}, \quad (24) \\ \arg\max \mu_{P_\circ}(x) = x^u_{\underline{\tilde{P}}} \,(= x^\ell_{\underline{\tilde{P}}}), & \text{otherwise}, \quad (25) \end{cases}$$

where $\mu_{P_\circ}(x)$ denotes the membership function of $P_\circ$. Eq. (25) is applied if facts are given by singletons, namely in the case where $x^u_{\underline{\tilde{P}}} = x^\ell_{\underline{\tilde{P}}}$. When $P_j, (j = 1,2,\ldots,n)$ are all defined by triangular membership functions, $\tilde{P}_\circ$ is obtained with a triangular membership function. In this case, $P_\circ$ can be easily generated by

$$\mu_{P_\circ}(x) = \mu_{\tilde{P}_\circ}(x + \Delta x), \quad \ldots\ldots \quad (26)$$

where

$$\Delta x = x^u_{(\underline{\tilde{P}}_\circ)_{\alpha'}} - x^u_{\underline{\tilde{P}}}$$
$$(= x^\ell_{(\underline{\tilde{P}}_\circ)_{\alpha'}} - x^\ell_{\underline{\tilde{P}}}),$$
$$(\tilde{P}_\circ)_{\alpha'} = [x^\ell_{(\tilde{P}_\circ)_{\alpha'}}, x^u_{(\tilde{P}_\circ)_{\alpha'}}],$$
$$\alpha' = 1 - \frac{w_{\underline{\tilde{P}}}}{w_{\underline{\tilde{P}}_\circ}}.$$

The symbol $(\tilde{P}_\circ)_{\alpha'}$ denotes the α-cut of $\tilde{P}_\circ$ at the level $\alpha = \alpha'$. Since the membership function of $\tilde{P}_\circ$ is triangular, $\tilde{P}_\circ$ is a convex fuzzy set. Therefore, each α-cut of $\tilde{P}_\circ$ is proved to be a closed interval.

Step 5c-3: Perform the selected inference method with "$P_j \Rightarrow Q_j$," $(j = 1,2,\ldots,n)$ and $\tilde{P}$ at $\omega(0) = \omega(1) = 1$. The deduced consequence is denoted by $Q_\circ$. When $Q_j, (j = 1,2,\ldots,n)$ are all defined by triangular membership functions, the membership function $\mu_{Q_\circ}(y)$ of $Q_\circ$ is proved to be triangular.

Step 6c: If $w_{\underline{\tilde{P}}} > w_{\underline{\tilde{P}}_\circ}$, proceed to Steps 6c-1 below:

Step 6c-1: Tune $\omega(0)$ in the selected inference method with FTRs, under the condition that $\omega(0) \geq 1$, so as to make $w_{\underline{\tilde{P}'}}$ equal to $w_{\underline{\tilde{P}}}$, where

$$w_{\underline{\tilde{P}'}} = x^u_{\underline{\tilde{P}'}} - x^\ell_{\underline{\tilde{P}'}}. \quad (27)$$

The symbols $x^u_{\underline{\tilde{P}'}}$ and $x^\ell_{\underline{\tilde{P}'}}$ represent the least upper and the greatest lower bounds of the support set $\underline{\tilde{P}'}$ of $\tilde{P}'$, respectively. The optimized value of $\omega(0)$ in the tuning is denoted by ω_0.

Step 6c-2: Perform the selected inference method with the FTRs and $\tilde{P}$ under the conditions that $\omega(1) = 1$ and $\omega(0) = \omega_0$. The deduced fuzzy set and its membership function are denoted by $\tilde{P}_\omega$ and $\mu_{\tilde{P}_\omega}(x)$, respectively. When $P_j, (j = 1,2,\ldots,n)$ are all defined by triangular membership functions, $\mu_{\tilde{P}_\omega}(x)$ is proved to be triangular.

Step 6c-3: If the selected inference method is either α-GEMII or α-GEMINAS for deducing consequences by the selected inference method itself, proceed to Step 6c-5; otherwise, follow Steps 6c-4 and 6c-5:

Step 6c-4: Generate a fuzzy set P_ω by shifting the position of $\tilde{P}_\omega$ so as to satisfy the following equation:

$$\underline{P}_\omega = \underline{\tilde{P}}, \quad (28)$$

where $\underline{P}_\omega$ denotes the support set of P_ω. Since P_ω and $\tilde{P}$ are convex, $\underline{\tilde{P}}_\omega$ and $\underline{\tilde{P}}$ are represented by closed intervals $[x^\ell_{\underline{\tilde{P}}_\omega}, x^u_{\underline{\tilde{P}}_\omega}]$ and $[x^\ell_{\underline{\tilde{P}}}, x^u_{\underline{\tilde{P}}}]$, respectively. The membership function $\mu_{P_\omega}(x)$ of P_ω is given by

$$\mu_{P_\omega}(x) = \mu_{\tilde{P}_\omega}(x + \Delta x), \quad (29)$$
$$\Delta x = x^u_{\underline{\tilde{P}}_\omega} - x^u_{\underline{\tilde{P}}}$$
$$(= x^\ell_{\underline{\tilde{P}}_\omega} - x^\ell_{\underline{\tilde{P}}}).$$

Step 6c-5: Perform the selected inference method with "$P_j \rightarrow Q_j$," $(j = 1,2,\ldots,n)$ and $\tilde{P}$ under the conditions that $\omega(1) = 1$ and $\omega(0) = \omega_0$. Let the deduced fuzzy set be denoted by Q_ω. When $Q_j, (j = 1,2,\ldots,n)$ are all defined by triangular membership functions, the membership function $\mu_{Q_\omega}(y)$ of Q_ω is proved to be triangular.

Step 7c: Generate a reference antecedent fuzzy set P_{ref} and a reference consequent fuzzy set Q_{ref} by using

$$P_{\text{ref}} = \begin{cases} P_\circ, & w_{\underline{\tilde{P}}} \leq w_{\underline{\tilde{P}}_\circ}, \\ P_\omega, & \text{otherwise}, \end{cases} \quad (30)$$

$$Q_{\text{ref}} = \begin{cases} Q_\circ, & w_{\underline{\tilde{P}}} \leq w_{\underline{\tilde{P}}_\circ}, \\ Q_\omega, & \text{otherwise}. \end{cases} \quad (31)$$

Let the membership functions of P_{ref} and Q_{ref} be denoted by $\mu_{P_{\text{ref}}}(x)$ and $\mu_{Q_{\text{ref}}}(y)$, respectively.

Step 8c: If the selected inference method is either α-GEMII or α-GEMINAS for deducing consequences by the selected inference method itself, the consequence $\tilde{Q}$ is given by Q_{ref}, namely $\tilde{Q} = Q_{\text{ref}}$, and proceed to Step 13c; otherwise, proceed to Step 9c below.

Step 9c: Generate a reference fuzzy rule "$P_{\text{ref}} \Rightarrow Q_{\text{ref}}$" by assigning P_{ref} and Q_{ref} to the antecedent and consequent parts of the reference fuzzy rule, respectively.

Step 10c: Perform the converse truth qualification (CTQ) between $\tilde{P}$ and P_{ref} by using

$$\tau_{\tilde{P}}(p) = \sup_{x = \mu^{-1}_{P_{\text{ref}}}(p)} \mu_{\tilde{P}}(x). \quad (32)$$

Let T_{ctq} denote the LTV whose membership function is given by $\tau_{\tilde{P}}(p)$. Note that CTQ by Eq. (32) can be efficiently performed via α-cuts as proved in [24].

Step 11c: Perform the compositional operation C for $\tau_{\tilde{P}}(p)$:

$$\tau_{\tilde{Q}}(q) = C[\tau_{\tilde{P}}(p),\ p \rightarrow q]. \quad (33)$$

When the max-min composition via truth spaces is adopted for C, $\tau_{\tilde{Q}}(q)$ is obtained by

$$\tau_{\tilde{Q}}(q) = \max_p \min[\tau_{\tilde{P}}(p),\ p \rightarrow q]. \quad (34)$$

Let T_{cmp} denote the LTV whose membership function is given by $\tau_{\tilde{Q}}(q)$. Note that the max-min composition by Eq. (34) can be efficiently conducted via α-cuts as proved in [24]. Moreover, CTQ by Eq. (32) and the max-min composition by Eq. (34) with either the Gaines-Rescher implication or the Gödel implication can be efficiently performed via α-cuts in their integrated form as proved in [16].

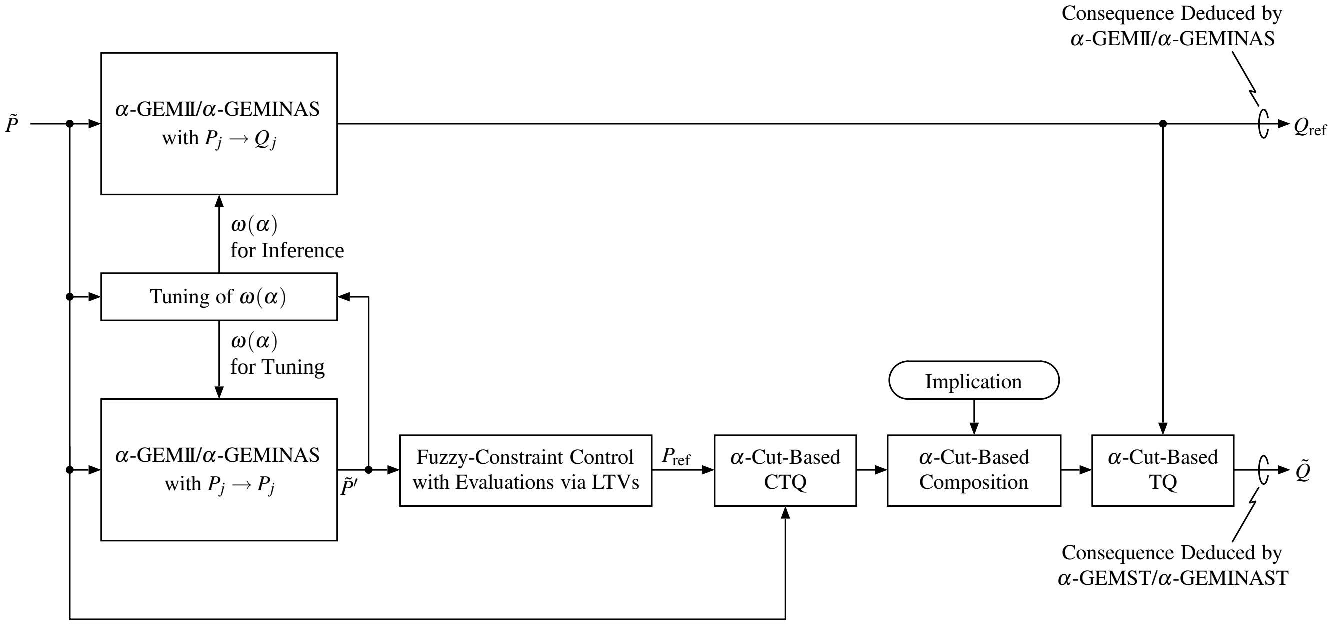

Fig. 4. The unified platform for the inference methods in the α-GEM family.

Step 12c: Perform the truth qualification (TQ) by using

$$\mu_{\tilde{Q}}(y) = \tau_{\tilde{Q}}(\mu_{Q_{\mathrm{ref}}}(y)), \quad \ldots \ldots \quad (35)$$

where $\mu_{\tilde{Q}}(y)$ gives the membership function of the consequence $\tilde{Q}$. Note that TQ by Eq. (35) can be efficiently performed via α-cuts as proved in [24].

Step 13c: Output $\tilde{Q}$ as the final consequence.

As shown in the operational steps, the fuzzy-constraint propagation is controlled at the multi-level of α in its α-cut-based operations.

Equation (25) is derived in this paper as a criterion in the fuzzy-constraint propagation control for facts given by singletons. On the other hand, Eq. (24) has been originally proposed for facts given by fuzzy sets in the sense of non-singletons where $x^u_{\underline{\tilde{P}}} \neq x^\ell_{\underline{\tilde{P}}}$ [16]. When facts are singletons, the equation $x^u_{\underline{\tilde{P}}} = x^\ell_{\underline{\tilde{P}}}$ holds. In this case, Eq. (24) does not uniquely determine the value of Δx because the relation $\mu_{P_\circ}(x^u_{\underline{\tilde{P}}}) = \mu_{P_\circ}(x^\ell_{\underline{\tilde{P}}})$ in Eq. (24) always holds at any values of $x^u_{\underline{\tilde{P}}}(= x^\ell_{\underline{\tilde{P}}})$. Eq. (25) is derived by the following process: As the difference $|\, x^u_{\underline{\tilde{P}}} - x^\ell_{\underline{\tilde{P}}} \,|\ (> 0)$ is decreased, $\mu_{P_\circ}(x^u_{\underline{\tilde{P}}})$ and $\mu_{P_\circ}(x^\ell_{\underline{\tilde{P}}})$ converge to $\max_x \mu_{P_\circ}(x)$ accordingly in applying Eq. (24) since $\mu_{P_\circ}(x)$ is unimodal. It means that Eq. (25) is derived by the convergence in Eq. (24). In this sense, Eq. (25) is in conformity with Eq. (24). The derivation of Eq. (24) is presented in [16].

Figure 4 depicts the unified platform proposed in this paper. As shown in the figure, consequences with α-GEMII and α-GEMINAS are obtained in the course of the operations for α-GEMST and α-GEMINAST, respectively.

7. Singleton-Consequent-Type Fuzzy Inference in the Schemes of Methods for Non-Sparse Rule Bases in the α-GEM Family

In many applications, singleton consequences (also referred to as singleton outputs) are often required with singleton facts (also referred to as singleton inputs). Singleton-consequent-type fuzzy inference is effective for treating systems with singleton inputs and outputs. It deduces singletons without defuzzification as shown in Eq. (15) and its operations are simple along with the singleton inputs and consequent singletons in fuzzy rules. It has been successfully applied to a wide variety of fields because of these features. This section derives the conditions to make α-GEMII and α-GEMST equivalent to singleton-consequent-type fuzzy inference. The conditions provide an effective way to make α-GEMII, α-GEMST, and singleton-consequent-type fuzzy inference transformed into each other in the unified platform proposed in Section 6, especially in learning for selecting the inference methods as well as for optimizing fuzzy rules.

7.1. Conditions of Equivalence Between α-GEMII and Singleton-Consequent-Type Fuzzy Inference

The following proposition shows the conditions to turn α-GEMII into singleton-consequent-type fuzzy inference:

Proposition 2 α-GEMII is equivalent to singleton-consequent-type fuzzy inference when the following conditions are satisfied:

Condition 1a: Facts are given by singletons.

Condition 2a: The consequent parts of fuzzy rules are defined by singletons.

■

Proof: The fuzzy-constraint propagation is controlled with the support-set widths of $\tilde{P}$ and $\tilde{P}'$ in this paper as described in Section 6. It means that the value of $\omega(0)$ controls the fuzzy-constraint propagation. Then, the following discussions focus on the support-set widths of $\tilde{P}$ and $\tilde{P}'$, wherein the optimization of $\omega(0)$ plays a central role.

The antecedent fuzzy sets $P_j, (j = 1,2,\ldots,n)$ for α-GEMII are not singletons and overlap each other to cover the entire area of the input space for given facts. Then, their support-set widths are more than zero. When given facts are singletons, their support-set widths are zero, namely their minimum value. In α-GEMII, the smaller values of $\omega(\alpha)$ decrease the widths of the α-cuts of $\tilde{P}'$ under the condition that $\omega(\alpha) \geq 1$ as shown in Relation (23). Thus, the fuzzy-constraint propagation control in Steps 1b–3b minimizes the support-set width of $\tilde{P}'$ to a large extent by decreasing the value of $\omega(0)$ since the support-set width of $\tilde{P}$ is zero. As a result, the value of $\omega(0)$ automatically becomes 1, namely its smallest value, because of the condition that $\omega(\alpha) \geq 1$ as shown in Relation (23).

When $\omega(0) = 1$, Eqs. (20) and (21) for α-GEMII are equivalent to Eqs. (12) and (13) for α-RITHM, respectively. When $Q_j, (j = 1,2,\ldots,n)$ in the consequent parts of fuzzy rules are given by singletons, Eqs. (12) and (13) for α-RITHM turn into Eq. (15) whose value is equivalent to the consequence deduced by singleton-consequent-type fuzzy inference as discussed in Section 4.1. Therefore, α-GEMII is equivalent to singleton-consequent-type fuzzy inference under Conditions 1a and 2a. Q.E.D.

7.2. Conditions of Equivalence Between α-GEMST and Singleton-Consequent-Type Fuzzy Inference

The following proposition presents the conditions to make α-GEMST equivalent to singleton-consequent-type fuzzy inference:

Proposition 3 α-GEMST is equivalent to singleton-consequent-type fuzzy inference when the following conditions are satisfied:

Condition 1b: Facts are given by singletons.

Condition 2b: The antecedent parts of fuzzy rules are defined by normal and unimodal membership functions.

Condition 3b: The consequent parts of fuzzy rules are defined by singletons.

Condition 4b: The max-t-norm composition is adopted and the truth function $p \to q$ of the implication $T_{p\to q}$ satisfies the following condition:

$$p \to q = \begin{cases} 1, & p = 1,\ q = 1, \\ 0, & p = 1,\ q = 0. \end{cases} \quad \ldots\ldots \quad (36)$$

■

Proof: Since reference fuzzy rules in α-GEMST are obtained with α-GEMII under Conditions 1b and 3b, Proposition 2 proves that the consequent parts of the reference fuzzy rules are given by singletons. Moreover, the consequent singletons of the reference fuzzy rules are equivalent to the consequences deduced by singleton-consequent-type fuzzy inference, according to Proposition 2. In order to make α-GEMST deduce the consequences equal to the consequent singletons, the following condition is required to be satisfied:

$$\tau_{\tilde{Q}}(q) = \begin{cases} 1, & q = 1, \\ 0, & q = 0. \end{cases} \quad \ldots\ldots\ldots\ldots \quad (37)$$

Eq. (37) makes TQ straightforwardly output the consequent singletons of the reference fuzzy rules as the final consequences deduced by α-GEMST. Namely, it proves that α-GEMST is equivalent to singleton-consequent-type fuzzy inference under Conditions 1b and 3b. Note that the values of $\tau_{\tilde{Q}}(q),\ q \in (0,1)$ are arbitrary as a condition of the equivalence. This is because the consequent parts of the reference fuzzy rules are given by singletons and thus their membership grades are either 0 or 1.

From Conditions 1b, 2b, and Eq. (25),

$$\mu_{P_\circ}(x_{\tilde{P}}) = 1, \quad \ldots\ldots\ldots\ldots\ldots \quad (38)$$

where $x_{\tilde{P}}$ denotes a numerical value given as a fact, namely

$$\mu_{\tilde{P}}(x) = \begin{cases} 1, & x = x_{\tilde{P}}, \\ 0, & \text{otherwise}, \end{cases} \quad \ldots\ldots\ldots \quad (39)$$

and then $x_{\tilde{P}} = x^u_{\underline{\tilde{P}}} = x^\ell_{\underline{\tilde{P}}}$. Eq. (30) provides $P_{\text{ref}} = P_\circ$ and thus the following equation is obtained by Eq. (38):

$$\mu_{P_{\text{ref}}}(x_{\tilde{P}}) = 1. \quad \ldots\ldots\ldots\ldots\ldots \quad (40)$$

From Eqs. (39) and (40), the following truth function is obtained by CTQ performed for $\mu_{P_{\text{ref}}}(x)$ and $\mu_{\tilde{P}}(x)$:

$$\tau_{\tilde{P}}(p) = \sup_{x=\mu^{-1}_{P_{\text{ref}}}(p)} \mu_{\tilde{P}}(x) = \begin{cases} 1, & p = 1, \\ 0, & \text{otherwise}. \end{cases} \quad . \ (41)$$

Since t-norm Ⓣ satisfies x Ⓣ $0 = 0$ and x Ⓣ $1 = x$, Eq. (41) provides

$$\begin{cases} \tau_{\tilde{P}}(p) \circledt (p \to q) = 1 \circledt (p \to q) = p \to q, & p = 1, \\ \tau_{\tilde{P}}(p) \circledt (p \to q) = 0 \circledt (p \to q) = 0, & \text{otherwise}. \end{cases} \quad \ldots\ldots \quad (42)$$

Thus, max-t-norm composition results in

$$\tau_{\tilde{Q}}(q) = \max\left[\tau_{\tilde{P}}(p) \circledt (p \to q)\right] = p \to q, \quad p = 1. \ (43)$$

This equation means that the result of the composition is determined only by $p \to q$ at $p = 1$. Eqs. (37) and (43) prove Eq. (36). Q.E.D.

It should be noted that many of the implications for the fuzzy logic satisfy Eq. (36). This is because the implications for the fuzzy logic are often created so as to include the implication for the two-valued logic and the implication for the two-valued logic itself satisfies Eq. (36). Moreover, $p \wedge q$ also satisfies Eq. (36), which has been often applied to a wide variety of fields, although it does not include the implication for the two-valued logic.

In Condition 4b, the max-t-norm composition is adopted for generalizing the condition to a large extent. When the min operation is used as a t-norm, namely, the max-min composition is applied in α-GEMST, the compositional operation can be performed independently at each level of α [24]. The independency contributes to the fast computation of α-GEMST with parallel processing for each level of α.

8. Singleton-Consequent-Type Fuzzy Inference in the Scheme of Methods for Sparse Rule Bases in the α-GEM Family

This section clarifies the conditions that make α-GEMINAS and α-GEMINAST equivalent to singleton-consequent-type fuzzy inference. α-GEMINAS and α-GEMINAST can deduce the same consequences that singleton-consequent-type fuzzy inference deduces, although they have been originally proposed for sparse rule bases. The clarified conditions provide an effective way to make α-GEMINAS, α-GEMINAST, and singleton-consequent-type fuzzy inference transformed into each other in the unified platform proposed in Section 6, especially in learning for selecting the inference methods as well as for optimizing fuzzy rules. For detailed discussions, activating points and activating-point interpolation are introduced first and then the conditions of the equivalence are derived.

8.1. Activating Points and Activating-Point Interpolation of Sparse Fuzzy Rules

Activating points are generated so as to make interpolated fuzzy rules activated by each given fact. Activating points are thus defined as follows [20]:

Definition 9 When antecedent fuzzy sets are defined in the universe of discourse X, *activating points* in X are given by

$$\{x \mid \tilde{p} > 0,\ x = x^{\circ}_{\hat{P}},\ x \in X\}, \qquad (44)$$

where $x^{\circ}_{\hat{P}}$ denotes the reference point of the antecedent fuzzy set $\hat{P}$ of an interpolated fuzzy rule and $\tilde{p}$ represents the compatibility degree between a given fact $\tilde{P}$ and $\hat{P}$. The fuzzy rules are interpolated in accordance with Steps 1d–5d shown later. ■

In the following, a method for fuzzy rule interpolation is introduced, which is named *activating-point interpolation* [20]. In this method, fuzzy rules are interpolated at a number of activating points. Activating-point interpolation is performed so as to make an activating point equal to the reference point of an antecedent fuzzy set in an interpolated fuzzy rule. Regarding this process, activating points are divided into two classes defined in the following proposition, where the activating points are denoted by $\hat{x}^{(i)}_{\tilde{P}}, (i = 1, 2, \ldots, N)$.

Definition 10 Activating points are marked with *effective* or *ineffective activating-points* as follows [20]: When $\hat{x}^{(i)}_{\tilde{P}}$ is equal to the reference point of an antecedent fuzzy set in an initially given fuzzy rule, it is marked with *an ineffective activating-point*. Otherwise, it is marked with *an effective activating-point*. ■

Activating-point interpolation is performed only with effective activating-points in order to avoid duplicating fuzzy rules by interpolation: This process prevents fuzzy rules, identical to the initially given fuzzy rules, from being generated by interpolation. For the following discussions, effective activating-points are denoted by $x^{(k)}_{\tilde{P}}, (k = 1, 2, \ldots, n')$. Moreover, let P_j be ordered so as to satisfy $x^{\circ}_{P_j} < x^{\circ}_{P_{j+1}}$, where $x^{\circ}_{P_j}$ and $x^{\circ}_{P_{j+1}}$ denote the reference points of P_j and P_{j+1}, respectively.

Inference with activating-point interpolation and α-GEMII is performed by the following steps [20]:

Step 1d: Determine P_j and P_{j+1} so as to satisfy $x^{\circ}_{P_j} < x^{(k)}_{\tilde{P}} < x^{\circ}_{P_{j+1}}$.

Step 2d: Calculate the Euclidean distances L_{kj} and $L_{k,j+1}$ defined as follows:

$$L_{kj} = |\, x^{(k)}_{\tilde{P}} - x^{\circ}_{P_j} \,|, \qquad (45)$$

$$L_{k,j+1} = |\, x^{(k)}_{\tilde{P}} - x^{\circ}_{P_{j+1}} \,|. \qquad (46)$$

Then, obtain $\tilde{p}_{kj}$ and $\tilde{p}_{k,j+1}$ by using

$$\tilde{p}_{kj} = \frac{|\, L_k - L_{kj} \,|}{L_k}, \qquad (47)$$

$$\tilde{p}_{k,j+1} = \frac{|\, L_k - L_{k,j+1} \,|}{L_k}, \qquad (48)$$

where $L_k = L_{kj} + L_{k,j+1}$. Let $\tilde{p}_{kj}$ and $\tilde{p}_{k,j+1}$ be called *distance-based compatibility degrees* for P_j and P_{j+1}, respectively. Repeat Steps 1d and 2d to obtain $\tilde{p}_{kj}$ and $\tilde{p}_{k,j+1}$ for all $k\,(= 1, 2, \ldots, n')$.

Step 3d: Deduce P'_k for all $k\,(= 1, 2, \ldots, n')$, using α-GEMII at $\omega(\alpha) = 1$ with the FTRs "$P_j \Rightarrow P_j$" and "$P_{j+1} \Rightarrow P_{j+1}$," where $x^{(k)}_{\tilde{P}}, (k = 1, 2, \ldots, n')$ are given as facts. The distance-based compatibility degrees $\tilde{p}_{kj}$ and $\tilde{p}_{k,j+1}$ are used in place of the compatibility degrees for P_j and P_{j+1}, respectively.

Step 4d: Deduce Q'_k for all $k\ (= 1, 2, \ldots, n')$, using α-GEMII at $\omega(\alpha) = 1$ with "$P_j \Rightarrow Q_j$" and "$P_{j+1} \Rightarrow Q_{j+1}$," where $x^{(k)}_{\tilde{P}}, (k = 1, 2, \ldots, n')$ are given as facts. The distance-based compatibility degrees $\tilde{p}_{kj}$ and $\tilde{p}_{k,j+1}$ are used in place of the compatibility degrees for P_j and P_{j+1}, respectively.

Step 5d: Deduce the consequence $\tilde{Q}$ on the basis of α-GEMII by using all of the initially given fuzzy rules "$P_j \Rightarrow Q_j$," $(j = 1,2,\ldots,n)$ together with all of the interpolated fuzzy rules "$P'_k \Rightarrow Q'_k$," $(k = 1,2,\ldots,n')$. When the initially given fuzzy rules are not activated due to the sparseness of the rule bases, deduce $\tilde{Q}$ on the basis of α-GEMII only by using all of the interpolated fuzzy rules "$P'_k \Rightarrow Q'_k$," $(k = 1,2,\ldots,n')$.

Steps 3d and 4d define *the activating-point interpolation*.

Activating-point interpolation with α-GEMII makes it possible to perform the nonlinear mapping of both membership-function positions and shapes in accordance with the initially given fuzzy rules, particularly in the case where a number of the fuzzy rules are activated in a sparse rule base. The nonlinear mapping by using fuzzy rule interpolation at a number of points is more precisely discussed in [20].

8.2. Conditions of Equivalence Between α-GEMINAS and Singleton-Consequent-Type Fuzzy Inference

α-GEMINAS is proved to be equivalent to singleton-consequent-type fuzzy inference under some conditions as shown in the following proposition:

Proposition 4 Suppose that the following conditions for α-GEMINAS are satisfied:

Condition 1c: Facts are given by singletons.

Condition 2c: The antecedent and consequent parts of fuzzy rules are defined by singletons.

α-GEMINAS under Conditions 1c and 2c is equivalent to singleton-consequent-type fuzzy inference that satisfies the following conditions: The antecedent parts are defined with a TSFP. The TSFP is constructed by using fuzzy sets whose core sets are equal to the antecedent singletons of the fuzzy rules for α-GEMINAS. The consequent singletons are the same as those in the fuzzy rules for α-GEMINAS. ■

Proof: Let $P_j, (j = 1,2,\ldots,n)$ and $\tilde{P}$ be given by numerical values $x_{P_j}, (j = 1,2,\ldots,n)$ and $x_{\tilde{P}}$, respectively, since P_j and $\tilde{P}$ are to be given by singletons according to Conditions 1c and 2c. Namely, their membership functions are respectively represented by

$$\mu_{P_j}(x) = \begin{cases} 1, & x = x_{P_j}, \\ 0, & \text{otherwise}, \end{cases} \quad (49)$$

$$\mu_{\tilde{P}}(x) = \begin{cases} 1, & x = x_{\tilde{P}}, \\ 0, & \text{otherwise}. \end{cases} \quad (50)$$

It should be noted that $P_j, (j = 1,2,\ldots,n)$ can be defined by singletons since α-GEMINAS has been originally proposed for sparse rule bases. Since $P_j, (j = 1,2,\ldots,n)$ and $\tilde{P}$ are all singletons, their support-set widths are zero. In this case, the activating point is only at $x = x_{\tilde{P}}$ and then α-GEMINAS generates only one fuzzy rule by interpolation, but not an infinite number of fuzzy rules. Moreover, the antecedent and consequent parts of the interpolated fuzzy rule are given by singletons because of the following reasons: α-GEMII is performed in Steps 3d and 4d for fuzzy rule interpolation, wherein distance-based compatibility degrees are obtained and are used as compatibility degrees. Then, α-GEMII can be regarded as being performed under the same conditions as in Proposition 2 which proves for α-GEMII to deduce singletons. As a result, the antecedent and consequent parts of interpolated fuzzy rules are given by singletons. Thereby, the antecedent and consequent parts of both the initially given and the interpolated fuzzy rules are represented with singletons. Therefore, one of these fuzzy rules is activated at $x = x_{\tilde{P}}$. Since the reference point of a singleton is equal to the element of the singleton,

$$x^{\circ}_{P_j} = x_{P_j}, \quad (51)$$

$$x^{\circ}_{\tilde{P}} = x_{\tilde{P}}, \quad (52)$$

where $x^{\circ}_{P_j}$ and $x^{\circ}_{\tilde{P}}$ denote the reference points of P_j and $\tilde{P}$, respectively.

Let the fuzzy sets for constructing the TSFP be denoted by $\check{P}_j, (j = 1,2,\ldots,n)$. Their core sets are respectively given by $x^{\circ}_{P_j}, (j = 1,2,\ldots,n)$ as stated in Proposition 4. The membership function of $\check{P}_j$ is denoted by $\mu_{\check{P}_j}(x)$. The consequent singleton of the fuzzy rule with $\check{P}_j$ is given by Q_j which is the consequent singleton in the fuzzy rule for α-GEMINAS. Suppose that $x_{P_j} \leq x_{\tilde{P}} \leq x_{P_{j+1}}$ for the following discussions.

When $x_{P_{j'}} = x_{\tilde{P}}, \exists j' \in \{1,2,\ldots,n\}$, α-GEMINAS deduces the consequence equal to $Q_{j'}$ because only one fuzzy rule, namely "$P_{j'} \Rightarrow Q_{j'}$," is activated at $x = x_{\tilde{P}} (= x_{P_{j'}})$. In this case, singleton-consequent-type fuzzy inference deduces the same consequence that α-GEMINAS deduces, according to Eq. (15), when it is performed with the TSFP stated in Proposition 4. This is because of the following reasons: The compatibility degree of $\check{P}_{j'}$ is 1 at $x = x_{\tilde{P}} (= x_{P_{j'}})$, and the other compatibility degrees of $\check{P}_j, (j \neq j')$ are 0 by the effect of the TSFP. As a result, only one fuzzy rule "$\check{P}_{j'} \Rightarrow Q_{j'}$" is activated at $x = x_{\tilde{P}} (= x_{P_{j'}})$ and therefore singleton-consequent-type fuzzy inference deduces the consequence that is equal to $Q_{j'}$ given by a singleton.

When $x_{P_j} \neq x_{\tilde{P}}, (j = 1,2,\ldots,n)$, there exists only one effective activating point $x^{(1)}_{\tilde{P}}$ at the value of $x_{\tilde{P}}$ in X. It leads to

$$x^{(1)}_{\tilde{P}} = x_{\tilde{P}} = x^{\circ}_{\tilde{P}}, \quad (53)$$

in accordance with Eq. (52). In this case, $\mu_{\check{P}_j}(x)$ in $[x^{\circ}_{P_j}, x^{\circ}_{P_{j+1}}]$ is given by

$$\mu_{\check{P}_j}(x) = \frac{x^{\circ}_{P_{j+1}} - x}{x^{\circ}_{P_{j+1}} - x^{\circ}_{P_j}} = \frac{x^{\circ}_{P_{j+1}} - x}{L_1}, \quad x \in [x^{\circ}_{P_j}, x^{\circ}_{P_{j+1}}]. \quad (54)$$

Table 1. Conditions of equivalence between the methods in the α-GEM family and singleton-consequent-type fuzzy inference.

Rule-Base Types	Inference Methods	Conditions of Equivalence			
		Antecedent Parts	Consequent Parts	Facts	Implications
Non-Sparse Rule Bases	α-GEMII	Fuzzy Sets	Singletons	Singletons	—
	α-GEMST	Fuzzy Sets	Singletons	Singletons	Eq. (36)
Sparse Rule Bases	α-GEMINAS	Singletons	Singletons	Singletons	—
	α-GEMINAST	Singletons	Singletons	Singletons	Eq. (36)

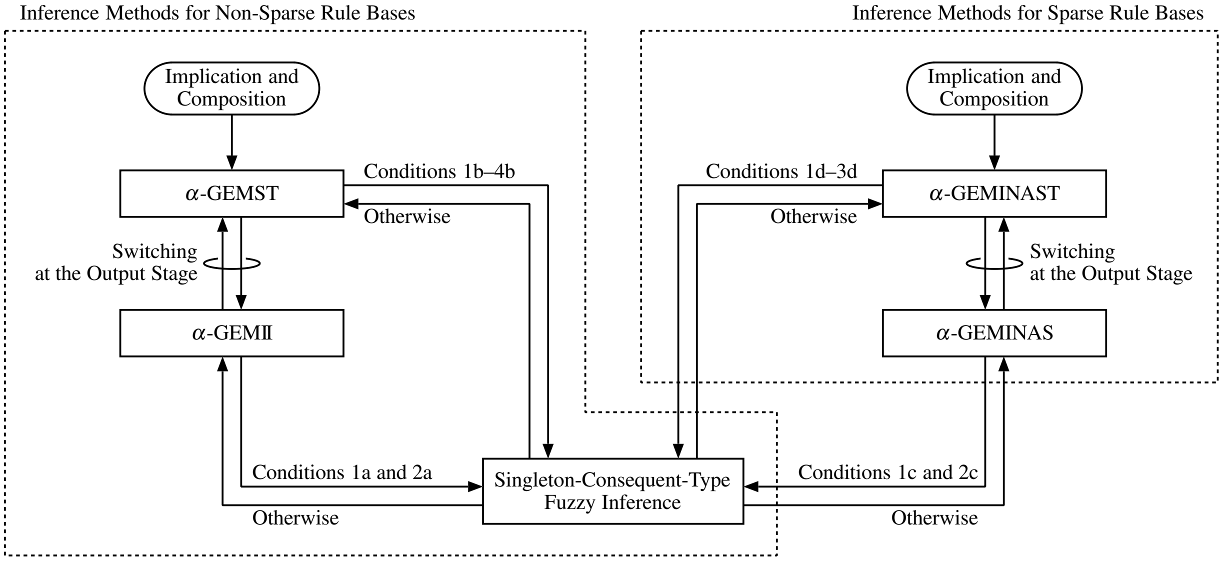

Fig. 5. Transformation between the inference methods in the α-GEM family and singleton-consequent-type fuzzy inference.

When $x_{\tilde{P}} \in [x^{\circ}_{P_j}, x^{\circ}_{P_{j+1}}]$, Eqs. (52)–(54) lead to

$$\mu_{\check{P}_j}(x_{\tilde{P}}) = \mu_{\check{P}_j}(x^{\circ}_{\tilde{P}}) = \mu_{\check{P}_j}(x^{(1)}_{\tilde{P}}) \quad (55)$$

$$= \frac{x^{\circ}_{P_{j+1}} - x^{(1)}_{\tilde{P}}}{L_1} \quad (56)$$

$$= \frac{|L_1 - L_{1j}|}{L_1} \quad (57)$$

$$= \tilde{p}_{1j}. \quad (58)$$

Eq. (58) is equivalent to Eq. (47) at $k = 1$. Therefore, it proves that the distance-based compatibility degree in α-GEMINAS is equal to the compatibility degree obtained from the TSFP. In the same way, the relation $\mu_{\check{P}_{j+1}}(x_{\tilde{P}}) = \tilde{p}_{1,j+1}$ can be proved. In this case, the operations in Step 4d to obtain Q'_1 are equivalent to those in α-GEMII performed with "$\check{P}_j \Rightarrow Q_j$," $(j = 1,2,\ldots,n)$ and the fact given by $x_{\tilde{P}}$, wherein $\omega(\alpha) = 1$ for all levels of α as stated in the step. Such operations in α-GEMII are equivalent to those in singleton-consequent-type fuzzy inference as can be found in the discussions in Section 4.3.1 since $Q_j, (j = 1,2,\ldots,n)$ are given by singletons. As a result, Q'_1 is a singleton and is equal to the consequence deduced by singleton-consequent-type fuzzy inference with the TSFP. Since there is only one effective activating point and thus there exists only one interpolated fuzzy rule given by "$P'_1 \Rightarrow Q'_1$," α-GEMINAS deduces the consequence equal to $\tilde{Q}'_1$. Therefore, α-GEMINAS is equivalent to singleton-consequent-type fuzzy inference with the TSFP under Conditions 1c and 2c. Q.E.D.

8.3. Conditions of Equivalence Between α-GEMINAST and Singleton-Consequent-Type Fuzzy Inference

The following proposition clarifies the conditions to make α-GEMINAST equivalent to singleton-consequent-type fuzzy inference:

Proposition 5 Suppose that the following conditions for α-GEMINAS are satisfied:

Condition 1d: Facts are given by singletons.

Condition 2d: The antecedent and consequent parts of fuzzy rules are defined by singletons.

Condition 3d: The max-t-norm composition is adopted and the truth function $p \rightarrow q$ of the implication $T_{p\rightarrow q}$ satisfies Eq. (36).

α-GEMINAST under Conditions 1d–3d is equivalent to singleton-consequent-type fuzzy inference that satisfies the following conditions: The antecedent parts are defined with a TSFP. The TSFP is constructed by using fuzzy sets whose core sets are equal to the antecedent singletons of the fuzzy rules for α-GEMINAST. The consequent singletons are the same as those in the fuzzy rules for α-GEMINAST. ■

Proof: Reference fuzzy rules are generated with α-GEMINAS in α-GEMINAST. When Conditions 1d and 2d are satisfied, Proposition 4 proves that the antecedent and consequent parts of the reference fuzzy rules are given by singletons. Proposition 4 also proves that the consequent singletons of the reference fuzzy rules are equivalent to the consequences deduced by singleton-consequent-type fuzzy inference with the TSFP. Condition 3d proves that α-GEMINAST deduces the consequences given by the consequent singletons of the reference fuzzy rules because the same discussions hold as in the proof of Proposition 3. Therefore, α-GEMINAST under Conditions 1d–3d is equivalent to singleton-consequent-type fuzzy inference with the TSFP. Q.E.D.

In Condition 3d, the max-t-norm composition is adopted for generalizing the condition to a large extent. When the min operation is used as a t-norm, namely, the max-min composition is applied in α-GEMINAST, the compositional operation can be performed independently at each level of α [24]. The independency contributes to the fast computation of α-GEMINAST with parallel processing for each level of α.

Table 1 summarizes the conditions of equivalence between the inference methods in the α-GEM family and singleton-consequent-type fuzzy inference. The conditions provide an effective way to make these inference methods transformed into each other in the unified platform, especially in learning for selecting the inference methods as well as for optimizing fuzzy rules. **Fig. 5** depicts the transformation between the inference methods.

9. Conclusion

The relations in properties and structures between the inference methods in the α-GEM family have been clarified first. Then, a unified platform has been proposed, especially for the most usable methods in the family at present. It is derived by the effective use of the above-mentioned relations. For the unified platform, a criterion is clarified for the fuzzy-constraint propagation control to uniquely determine the value of a parameter for facts given by singletons. Moreover, conditions are derived for equivalence between the methods in the α-GEM family and singleton-consequent-type fuzzy inference which has been practically applied to a wide variety of fields. Thereby, the unified platform contributes to the implementation of an inference engine for performing these inference methods in a single base.

All of the methods in the α-GEM family have the common properties: They mathematically prove the convexity of deduced consequences and thereby the consequences can be treated as fuzzy numerical values. They suppress the excessively large fuzziness and excessively small specificity of deduced consequences. Their operations can be performed independently at each level of α. The independency contributes to their fast computations with parallel processing for operations at each level of α.

α-GEM has been proposed first on the basis of α-cuts and the generalized mean. Then, α-GEMII has been proposed, which can prove the symmetricity of deduced consequences under some axiomatically derived conditions. α-GEMII has led to other inference methods because of its effectiveness in nonlinear mapping between convex fuzzy sets in accordance with fuzzy rules.

In non-sparse rule bases, α-GEMS and α-GEMST can be performed in synergy between α-GEMII and CRI. They provide various fuzzy-constraint propagation by selecting one pair from the widely available implications and compositional operations, while they can perform a nonlinear mapping between convex fuzzy sets. In sparse rule bases, α-GEMMI, α-GEMMIET, and α-GEMILIE can be performed, which are originated from the concept of multi-level interpolation. α-GEMILIE leads to α-GEMINAS which deduces consequences by using α-GEMII with fuzzy rules interpolated at an infinite number of activating points. α-GEMINAST has been proposed in synergy between α-GEMINAS and CRI. It can control the fuzzy-constraint propagation in various ways by arranging the support-set widths of antecedent fuzzy sets independently of the number of fuzzy rules and of the positions of the antecedent fuzzy sets in the input space. It also provides various fuzzy-constraint propagation by selecting one pair from the widely available implications and compositional operations while they can perform a nonlinear mapping between convex fuzzy sets.

A unified platform has been proposed in this paper, especially for α-GEMII, α-GEMST, α-GEMINAS, and α-GEMINAST which are most usable in practice at present. It is derived by the effective use of the relations in properties and structures between these inference methods in the α-GEM family. For the unified platform, a criterion is clarified for the fuzzy-constraint propagation control to uniquely determine the value of a parameter for facts given by singletons. Moreover, conditions have been derived to prove the equivalence between the methods in the α-GEM family and singleton-consequent-type fuzzy inference which has been conventionally applied to many fields. Thereby, the unified platform can contribute to the construction of an inference engine for both the inference methods in the α-GEM family and singleton-consequent-type fuzzy inference. It provides an effective way to transform the inference methods into each other in a single inference engine, especially in learning for selecting the inference methods as well as for optimizing fuzzy rules.

The authors make the α-GEM family grow toward other related methods and their applications. The discussions in this paper are expected to contribute to creating inference methods based on α-cuts and generalized mean.

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