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Paper:

Mixed Oligopoly: Analysis of Consistent Equilibria

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In this paper, a model of mixed oligopoly with conjectured variations equilibrium (CVE) is examined, in which one of the agents maximizes a convex combination of its net profit with the domestic social surplus. The agents' conjectures concern the price variations, which depend on the variations in their production outputs. Using the established existence and uniqueness results for the CVE (the *exterior equilibrium*) for any fixed set of feasible conjectures, the notion of the interior equilibrium is introduced by developing a conjecture consistency criterion. Then, the existence theorem for the interior equilibrium (defined as a CVE state with *consistent conjectures*) is proven. When the convex combination coefficient tends to 1 (thus transforming the model into the mixed oligopoly in its extreme form), two trends are apparent. First, for private companies, the equilibrium with consistent conjectures becomes more proficient than the Cournot-Nash equilibrium. Second, there exists a (unique) value of the convex combination coefficient such that the private agent's aggregate profit is the same in both the above-mentioned equilibria, which makes subsidies to producers or consumers unnecessary.

Keywords: management engineering, game theory, equilibrium theory

1. Introduction

Models of mixed oligopolies have attracted increased attention in the past decade and have become very popular in the literature. In contrast to a classical oligopoly, a mixed oligopoly usually includes at least one special agent in addition to the regular participants, who maximize their net profits. This special company strives to increase the value of an objective function that is different from the net profit. Many models of this kind include an agent that maximizes domestic social surplus (*cf.* [1–5]). An income-per-worker function substitutes the standard net profit objective function in some other sources (*cf.* [6–9]). Papers [10] and [11] examine a third kind of mixed duopoly with an extraordinary agent that aims to increase a convex combination of its net profit and the domestic social surplus.

Most of the above-mentioned works study mixed oligopoly with the classical Cournot, Hotelling, or Stackelberg approaches. Recently, the *conjectural variations equilibrium* (CVE) introduced by Bowley [12] and Frisch [13], another possible solution for static games, has received increased study. This concept supposes that the players behave as follows: each agent selects his/her optimal strategy under the assumption that every rival's response is a *conjectural variation function* of her/his own move. For example, as Laitner [14, p. 643] states, "Although the firms make their output decisions simultaneously, plan changes are always possible before production begins." In other words, in contrast to the Cournot-Nash framework, here, every firm supposes that its choice of output level will affect its opponents' behavior. The generated anticipation (or conjectural variation) function composes the core of the conjectural variations approach to decision making, or the CVE.

According to [15] and [16], the CVE has been the topic of various theoretical discussions (*cf.* [17]). Economists have made extensive use of various forms of the CVE to forecast the effects of non-cooperative behavior in nu-

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merous areas of economics. However, the literature on conjectural variations has focused mainly on two-player games because a serious conceptual difficulty arises if the number of agents is greater than two (*cf.* [15] and [18]).

In order to cope with this conceptual impediment in many-player games, Bulavsky [19] developed a strong new approach. Rather than assuming the equivalence (symmetry) of players in the oligopoly, he supposed that every player does *not* make conjectures redarding the (optimal) response functions of each of the other players, but only about the variations of the market clearing price, which depend on (infinitesimal) variations in the agent's output volume. Knowing their rivals' conjectures (called *influence coefficients*), every agent applies a verification procedure and determines whether his/her influence coefficient is *consistent* with those of the other players.

In the recent papers [18] and [20], the results of [19] were extended to the mixed duopoly and oligopoly cases, respectively. In both papers, the concept of exterior equilibrium was defined as the CVE with influence coefficients fixed in an exogenous mode. The existence and uniqueness theorems for this sort of CVE were established to be used as a keystone for the concept of interior equilibrium, which is the exterior equilibrium with consistent conjectures (influence coefficients). The consistency criteria, verification procedures, and existence theorems for the interior equilibrium were formulated and proven in [18] and [20].

In [22], the above-described theoretical outcomes were amplified to the case of a partially mixed oligopoly, in which the public company, similar to [10] and [11], maximizes a *convex combination* of its net profit and the domestic social surplus. The results of numerical experiments with a test model of an electricity market from [23], both with and without a public company among the agents, showed the importance of the CVE for the consumer. In this paper, we explore this importance in greater detail. When the convex combination coefficient tends to 1, thus pushing the model towards the standard mixed oligopoly, two interesting trends can be discerned. First, for private companies, the equilibrium with consistent conjectures becomes more proficient than the Cournot-Nash equilibrium. Second, there exists a (unique) value of the combination coefficient such that the private agents' aggregate profit is the same in both the above-mentioned equilibria (Cournot-Nash and Bulavsky), which makes subsidies from the authoritites to producers or consumers unnecessary.

The rest of this paper is arranged as follows. Section 2 specifies the model and the two kinds of equilibrium we consider (exterior and interior). In Section 3, we present the main theorem for the existence and uniqueness of the exterior equilibrium for any set of feasible conjectures (influence coefficients) as well as the formulas for the derivative of the equilibrium price p with respect to the active demand variable D. Section 4 introduces the consistency criterion and the definition of the interior equilibrium (which can be treated as a consistent CVE, or CCVE); the CCVE existence theorem from [22] is also discussed. To provide tools for future research concerning the interrelationships between the demand structure (with a demand function that is not necessarily smooth) and the CVEs with consistent conjectures (influence coefficients), the behavior of the latter as functions of a certain parameter governed by the derivative by p of the demand function is considered in Theorem 4.9. Section 5 deals with an important case of a linear demand function for duopoly and, more generally, oligopoly. In Section 6, a qualitative analysis of the results of the numerical experiments from [22] is developed, while concluding remarks are given in Section 7.

2. Mixed Oligopoly with Combined Payoff Functions

Following [22], consider a market with at least three producers of a homogeneous good with the cost functions $f_i(q_i)$, $i = 0, 1, \ldots, n$, $n \geq 2$, where q_i is the output of producer i. Consumers demand is described by a demand function $G(p)$, where p is the market price proposed by the producers. The value of active demand D is nonnegative and does not depend on the price. We fix the equilibrium between the demand and supply for a given price p by the following balance equality

$$\sum_{i=0}^{n} q_i = G(p) + D. \qquad (1)$$

We also assume the following properties of the model's data.

A1. The demand function $G = G(p) \geq 0$ defined over $p \in [0, +\infty)$ is non-increasing and continuously differentiable.

A2. For each agent $i = 0, 1, \ldots, n$, the cost function $f_i = f_i(q_i)$ is quadratic:

$$f_i(q_i) = \frac{1}{2} a_i q_i^2 + b_i q_i, \qquad (2)$$

where $a_i > 0, b_i > 0$, $i = 0, 1, \ldots, n$. In addition, we assume that

$$b_0 \leq \max_{1 \leq i \leq n} b_i. \qquad (3)$$

Remark 2.1 Although the assumption of $a_i > 0$, $i = 0, 1, \ldots, n$, may seem to be unacceptable in view of the scale effect often observed in real-life production economies, it is not uncommon in theories of classical and mixed oligopolies (*cf.* [3–5] and [23]). In the majority of cases, this assumption is the easiest way to ensure the concavity of each player's payoff function. However, this condition can be somewhat relaxed, as, for example, in [24], where the second derivative of the cost function is not assumed to be positive. Then, the desired payoff function's concavity is achieved by another assumption that combines the first derivative of the demand function and the second derivative of the cost function. Finally, the scale effect can also be modeled by permitting the first-order coefficients b_i, $i = 0, \ldots, n$ to be negative. We have already obtained the corresponding results for this more general case, and they will be published elsewhere. ■

Each private producer $i = 1,\dots,n$ chooses its output volume $q_i \geq 0$ so as to maximize its profit function $\pi_i(p,q_i) = p \cdot q_i - f_i(q_i)$. On the other hand, the public company with index $i = 0$ produces $q_0 \geq 0$ so as to maximize a convex combination of its profit and the domestic social surplus (defined as the difference between the consumer surplus, the private companies' aggregate revenue, and the public firm's production costs):

$$S(p;q_0,\dots,q_n) = \beta\left[\int\limits_0^{\sum_{i=0}^n q_i} p(x)dx - p\left(\sum_{i=1}^n q_i\right) - b_0q_0 - \frac{1}{2}a_0q_0^2\right] + (1-\beta)\left(pq_0 - b_0q_0 - \frac{1}{2}a_0q_0^2\right), \quad (4)$$

where $0 < \beta \leq 1$ (here, we follow [10] and [11]). We postulate that the agents (both public and private) assume that their choice of production volumes may affect the price value p. This assumption can be defined by the conjectured dependence of the price p on the output values q_i. Then, the first-order maximum condition to describe the equilibrium would have the following form for the public company ($i = 0$)

$$\frac{\partial S}{\partial q_0} = p - \left[(\beta-1)q_0 + \beta\sum_{i=1}^n q_i\right]\frac{\partial p}{\partial q_0} - f_0'(q_0)\begin{cases} = 0, & \text{if } q_0 > 0; \\ \leq 0, & \text{if } q_0 = 0; \end{cases} \quad (5)$$

and

$$\frac{\partial \pi_i}{\partial q_i} = p + q_i\frac{\partial p}{\partial q_i} - f_i'(q_i)\begin{cases} = 0, & \text{if } q_i > 0; \\ \leq 0, & \text{if } q_i = 0, \end{cases} \quad (6)$$

for private firm i ($i = 1,\dots,n$).

Thus, we see that to describe the agent's behavior, we must evaluate the behavior of the derivative $\partial p/\partial q_i = -v_i$ rather than the dependence of p on q_i, $i = 0,\dots,n$. Here, we introduce the negative sign (minus) to deal with nonnegative values of v_i. Of course, the conjectured dependence of p on q_i must provide (at least the local) concavity of the i-th agent's conjectured profit as a function of its output.

For instance, it is sufficient to assume coefficient v_i (henceforth referred to as the i-th agent's *influence coefficient*) to be nonnegative and constant. Then, the conjectured local dependence of private firm i's profit variation on the variation in production output $(\eta_i - q_i)$ has the form $[p - v_i(\eta_i - q_i)]\,\eta_i - pq_i - f_i(\eta_i) + f_i(q_i)$, which implies that the profit is a concave function with respect to η_i. Therefore, the profit's maximum condition at $\eta_i = q_i$, $i = 1,\dots,n$, is provided by the relationships

$$\begin{cases} p = v_iq_i + b_1 + a_iq_i, & \text{if } q_i > 0; \\ p \leq b_i, & \text{if } q_i = 0. \end{cases} \quad (7)$$

Similarly, the public company conjectures the local dependence of the domestic social surplus on its production output's variation η_0 in the form

$$\beta\left\{\int\limits_{q_0+\sum_{i=1}^n q_i}^{\eta_0+\sum_{i=1}^n q_i} p(x)dx - [p - v_0(\eta_0 - q_0)] \times \sum_{i=1}^n q_i\right\} + (1-\beta)\left\{[p - v_0(\eta_0 - q_0)]\,\eta_0 - pq_0\right\} - f_0(\eta_0) + f_0(q_0), \quad (8)$$

which indicates that the public company's payoff function is concave with respect to η_0; hence, the maximum condition at $\eta_0 = q_0$ can be written as follows:

$$\begin{cases} p = -\beta v_0\sum_{i=1}^n q_i + (1-\beta)v_0q_0 + b_0 + a_0q_0, & \text{if } q_0 > 0; \\ p \leq -\beta v_0\sum_{i=1}^n q_i + b_0, & \text{if } q_0 = 0. \end{cases} \quad (9)$$

If the agents' conjectures about the model were given exogenously, as assumed in [25] and [26], we would allow the values v_i to be functions of q_i and p. However, we use the approach from [18], [19], and [22], where the conjecture parameters for the equilibrium are determined simultaneously with the values for price p and the outputs q_i by a special verification procedure. In the latter case, the influence coefficients are scalar parameters determined only for the equilibrium. In Section 4, such an equilibrium state is referred to as an *interior* one, which is described by the set of variables and parameters $(p,q_0,\dots,q_n,v_0,\dots,v_n)$.

As mentioned in Remark 2.1, we have already obtained results concerning the models featuring so-called effects of scale (scale effects, economies of scale, etc.). Economies of scale are the cost advantages that enterprises obtain because of their size, output, or scale of operation, with the cost per unit of output generally decreasing as scale increases because fixed costs are spread out over more units of output. Often, operational efficiency is also greater with increased scale, which leads to lower variable costs as well. In our model, this effect can be revealed if the values of b_i are negative. In order to allow this effect, we must modify assumption **A2** as follows.

A2′. For all agents $i = 0,\dots,n$, their cost functions $f_i = f_i(q_i)$ are quadratic:

$$f_i(q_i) = \frac{1}{2}a_iq_i^2 + b_iq_i + c_i, \quad (10)$$

where $a_i > 0, b_i < 0$, $c_i > 0$, $i = 0,1,\dots,n$. In addition, we suppose that

$$b_i^2 - 4a_ic_i < 0, \quad i = 0,\dots,n, \quad (11)$$

and

$$G(0)+\sum_{i=0}^{n}\frac{b_i}{a_i}>0. \qquad (12)$$

Remark 2.2 In contrast to assumption **A2**, here, we need to suppose a fixed cost c_i to be positive for all producers $i=0,\dots,n$. Moreover, the technical condition (11) is needed to guarantee that the quadratic cost $f_i=f_i(q_i)$, $i=0,\dots,n$, never drops to zero for non-negative supply volumes $q_i\geq 0$, $i=0,\dots,n$. Finally, together with assumption **A1**, the strict inequality (12) provides for the existence of a unique solution for the balance equality (1) and optimality conditions (7) and (9) (i.e., the exterior equilibrium) for any nonnegative values of D,v_i, $i=1,\dots,n$, and all $v_0\in[0,\bar{u}_0)$, where

$$0<\bar{u}_0:=-a_0\left[\frac{G(0)+\sum\limits_{i=0}^{n}\frac{b_i}{a_i}}{\sum\limits_{i=1}^{n}\frac{b_i}{a_i}}\right]. \qquad (13)$$

In the special case of duopoly ($n=1$), the parameter $\bar{u}_0$ is defined with a simpler formula:

$$0<\bar{u}_0:=-\frac{a_0a_1}{b_1}\left[G(0)+\frac{b_0}{a_0}+\frac{b_1}{a_1}\right]. \qquad (14)$$

■

It is notable that all the properties and optimality conditions stated above are also valid for economies of scale assumptions. Moreover, optimality conditions (7) and (9) become even simpler because under **A2′**, zero output is not possible for any company. These and other differences that appear in the existence and uniqueness theorems for the exterior and interior equilibria will be discussed in Sections 3 and 4.

3. Exterior Equilibrium in Oligopoly

In order to present the verification procedure, we must study first another notion of equilibrium, the *exterior equilibrium* (*cf.* [18] and [20]), with parameters v_i given *exogenously*. The point $(p,q_0,\dots,q_n)$ is called an *exterior equilibrium state* for the given influence coefficients $(v_0,\dots,v_n)$, if the market is balanced, that is, if equality (1) is valid and the maximum conditions (7) and (9) hold.

In the following, we consider only the case in which the collection of producing participants is fixed (i.e., it does not depend on the values v_i of the influence coefficients). To guarantee this property, we make the following assumption.

A3. For the price $p_0=\max\limits_{1\leq j\leq n}b_j$, the following estimate holds

$$\sum_{i=0}^{n}\frac{p_0-b_i}{a_i}<G(p_0). \qquad (15)$$

The latter assumption, together with assumptions **A1** and **A2**, guarantees that for all nonnegative values of v_i, $i=1,\dots,n$, and for $v_0\in[0,\bar{v}_0)$, where $\bar{v}_0>0$ and

$$\bar{v}_0=a_0\left[\frac{G(p_0)-\frac{p_0-b_0}{a_0}}{\sum\limits_{i=1}^{n}\frac{p_0-b_i}{a_i}}-1\right] \qquad (16)$$

if

$$\sum_{i=1}^{n}\frac{p_0-b_i}{a_i}>0,$$

and $\bar{v}_0=+\infty$ otherwise, there always exists a unique solution of optimality conditions (7) and (9) satisfying balance equality (1) — that is, it is an exterior equilibrium state. Lemma 3.1 establishes that conditions (1), (7) and (9) can hold simultaneously if and only if $p>p_0$: that is, if and only if all outputs q_i, $i=0,\dots,n$ are strictly positive. The latter equivalence is demonstrated below (this proof appears in [22]: we repeat it for self-consistency).

Lemma 3.1. *Let assumptions* **A1**–**A3** *be valid. If a vector* $(p,q_0,q_1,\dots,q_n)$ *is an exterior equilibrium state, then the relationship* $p>p_0$ *is equivalent to the fact that all* $q_i>0$, $i=0,1,\dots,n$.

Proof. If a vector $(p,q_0,q_1,\dots,q_n)$ is an exterior equilibrium state then conditions (1), (7) and (9) hold simultaneously. In this case $p>p_0$ is equivalent to the fact that all outputs q_i are strictly positive, $i=0,1,\dots,n$. Indeed, if $p>p_0$, it is evident from (3) and (9) that neither the inequalities $p\leq b_i$, $i=1,\dots,n$ seen in (7), nor $p\leq\beta v_0\sum\limits_{i=1}^{n}q_i+b_0$ in (9) are possible, which means that none of q_i, $i=0,1,\dots,n$ can be zero. Conversely, if all $q_i>0$, $i=0,1,\dots,n$, it is straightforward from condition (7) that

$$p=v_iq_i+b_i+a_iq_i>b_i, \quad i=1,\dots,n,$$

hence $p>\max\limits_{1\leq j\leq n}b_j=p_0$. ■

We are now in a position to formulate the main result of this section. We have demonstrated the following theorem in [22], and the details of its very long proof are available from the authors upon request.

Theorem 3.2. *Under assumptions* **A1**–**A3**, *for any* $D\geq 0$, $v_i\geq 0$, $i=1,\dots,n$, *and* $v_0\in[0,\bar{v}_0)$, *there exists uniquely an exterior equilibrium state* $(p,q_0,q_1,\dots,q_n)$ *that depends continuously on the parameters* $(D,v_0,v_1,\dots,v_n)$. *The equilibrium price* $p=p(D,v_0,v_1,\dots,v_n)$ *as a function of these parameters is differentiable with respect to both* D *and* v_i, $i=0,1,\dots,n$. *Moreover,* $p(D,v_0,v_1,\dots,v_n)>p_0$, *and*

$$\frac{\partial p}{\partial D}=\frac{1}{F(\beta,a,v,p)}, \qquad (17)$$

with

$$F(\beta,a,v,p) = \frac{1}{a_0+(1-\beta)v_0} + \frac{a_0+v_0}{a_0+(1-\beta)v_0}\sum_{j=1}^{n}\frac{1}{a_j+v_j} - G'(p). \quad (18)$$

First, we consider the case of $n = 1$, that is, of the duopoly model.

3.1. Exterior Equilibrium in Duopoly with Economies of Scale

As mentioned above, in the economies of scale models, under assumption **A2′**, no company may supply zero output at equilibrium. Indeed, since the coefficients b_0 and b_1 are now both supposed to be *negative* in sign, optimality conditions (7) and (9) can hold *only* if $q_i > 0$, $i = 0,1$. More precisely, they reduce to the optimality conditions

$$p = v_1q_1 + b_1 + a_1q_1, \quad (19)$$

and

$$p = -\beta v_0q_1 + (1-\beta)v_0q_0 + b_0 + a_0q_0, \quad (20)$$

respectively.

Thus, the following key result of this subsection is easier to establish than Theorem 3.2. However, since its proof is too long, it will be published elsewhere.

Theorem 3.3. *Under assumptions* **A1** *and* **A2′***, for any* $D \geq 0$*,* $v_1 \geq 0$*, and* $v_0 \in [0,\bar{u}_0)$*, where* $\bar{u}_0$ *is as defined in* (14)*, there exists uniquely the exterior equilibrium* (p,q_0,q_1) *that depends continuously on the parameters* (D,v_0,v_1)*. The equilibrium price* $p = p(D,v_0,v_1)$ *as a function of these parameters is differentiable with respect to both D and* v_0,v_1*. Moreover,* $q_i > 0$*,* $i = 0,1$ *and*

$$\frac{\partial p}{\partial D} = \frac{1}{F(\beta,a,v,p)} \quad (21)$$

where

$$F(\beta,a,v,p) = \frac{1}{a_0+(1-\beta)v_0} + \frac{v_0+a_0}{a_0+(1-\beta)v_0}\cdot\frac{1}{v_1+a_1} - G'(p). \quad (22)$$

■

3.2. Exterior Equilibrium in Oligopoly with Economies of Scale

Similar to the duopoly case studied above, in the economies of scale models, under assumption **A2′**, we are assured that no company may supply zero output in the equilibrium. Indeed, since all coefficients b_i, $i = 0,1,\ldots,n$, are now *negative* in sign, optimality conditions (1), (7) and (9) can hold *only* if $q_i > 0$, $i = 0,1,\ldots,n$. More precisely, (7) and (9) reduce to the optimality conditions

$$p = v_iq_i + b_i + a_iq_i, \qquad i = 1,\ldots,n, \quad (23)$$

and

$$p = -\beta v_0\sum_{i=1}^{n} q_i + (1-\beta)v_0q_0 + b_0 + a_0q_0, \quad (24)$$

respectively.

Therefore, the following key result of this subsection is easier to establish than Theorem 3.2. However, since its proof is extremely long, it will be published elsewhere.

Theorem 3.4. *Under assumptions* **A1** *and* **A2′***, for any* $D \geq 0$*,* $v_i \geq 0$*,* $i = 1,\ldots,n$*, and* $v_0 \in [0,\bar{u}_0)$*, where* $\bar{u}_0$ *is as defined in* (13)*, there exists uniquely the exterior equilibrium* $(p,q_0,\ldots,q_n)$ *that depends continuously on the parameters* $(D,v_0,\ldots,v_n)$*. The equilibrium price* $p = p(D,v_0,\ldots,v_n)$ *as a function of these parameters is differentiable with respect to both D and* $v_0,\ldots,v_n$*. Moreover,* $q_i > 0$*,* $i = 0,\ldots,n$ *and*

$$\frac{\partial p}{\partial D} = \frac{1}{F(\beta,a,v,p)} \quad (25)$$

where

$$F(\beta,a,v,p) = \frac{1}{a_0+(1-\beta)v_0} + \frac{a_0+v_0}{a_0+(1-\beta)v_0}\sum_{j=1}^{n}\frac{1}{a_j+v_j} - G'(p). \quad (26)$$

4. Interior Equilibrium in Oligopoly

Again, following [22], we now define the interior equilibrium.We first describe the procedure for verifying the influence coefficients v_i as given in [22]. Assume that we have an exterior equilibrium state $(p,q_0,\ldots,q_n)$ that has occurred for some $v_0,\ldots,v_n$, and D. One of the producers, say, k, temporarily changes its behavior: it abstains from maximizing the conjectured profit (or the combination of the domestic social surplus and its net profit, as in case $k = 0$) and exercises small fluctuations in its output volume q_k. In mathematical terms, it is tantamount to restricting the model to the oligopoly of the agents i with $i \neq k$ with output q_k subtracted from the active demand.

A fluctuation in the supply by agent k is then equivalent to accepting the active demand varied by $\delta D_k := \delta(D-q_k) \equiv -\delta q_k$. If we consider these variations as infinitesimal, we conclude that by observing the corresponding variations of the equilibrium price, agent k can estimate the derivative of the equilibrium price with respect to the active demand, which *coincides* with agent k's own influence coefficient.

Applying Eqs. (17) and (18) from Theorem 3.2 (or Eqs. (21) and (22) from Theorem 3.3 or Eqs. (25) and (26) from Theorem 3.4, for the model with economies of scale) to calculate the derivatives, one must remember that agent k is (temporarily) absent from the equilibrium model; hence, terms with number $i = k$ must be eliminated from all the sums. Bearing this in mind, we have the following criterion.

4.1. Consistency Criterion

In the exterior equilibrium state $(p, q_0, \ldots, q_n)$, the influence coefficients v_k, $k = 0, \ldots, n$ are referred to as *consistent* if the following equalities hold:

$$v_0 = \frac{1}{\displaystyle\sum_{i=1}^{n} \frac{1}{a_i + v_i} - G'(p)}, \quad \ldots \ldots \ldots \quad (27)$$

and

$$v_i = \frac{1}{\displaystyle\frac{a_0 + v_0}{a_0 + (1-\beta)v_0} \sum_{j \neq i, j=1}^{n} \frac{1}{a_j + v_j} - G'(p)}, \quad . \quad (28)$$

$i = 1, \ldots, n$.

We are now ready to define the concept of interior equilibrium.

Definition 4.1. *The collection* $(p, q_0, q_1, \ldots, q_n, v_0, v_1, \ldots, v_n)$,*where* $v_k \geq 0$, $k = 0, 1, \ldots, n$, *is referred to as an interior equilibrium state if, for the considered influence coefficients, the collection* $(p, q_0, q_1, \ldots, q_n)$ *is an exterior equilibrium state, and the consistency criterion is satisfied for all* v_k, $k = 0, 1, \ldots, n$.

Remark 4.2. If all agents $i = 0, 1, \ldots, n$, were net profit-maximizing companies, Eqs. (27) and (28) would be reduced to the uniform ones obtained independently in [19] and [23]:

$$v_i = \frac{1}{\displaystyle\sum_{j \neq i, j=0}^{n} \frac{1}{a_j + v_j} - G'(p)}, \quad i = 0, 1, \ldots, n. \quad \ldots \ldots \quad (29)$$

■

The following theorem is an extension of Theorem 4.2 from [18] to the case of a mixed oligopoly with the combined payoff function.

Theorem 4.3. *Under assumptions* **A1**–**A3**, *for any* $D \geq 0$, *there exists an interior equilibrium state.*

Proof. The proof is an evident extension of that of Theorem 4.2 in [18]. ■

Remark 4.3. The only difference of Theorem 4.3 from the corresponding existence result for the interior equilibrium in an economies-of-scale mixed oligopoly model is that in the latter, assumption **A2** is replaced with assumption **A2′**, while assumption **A3** becomes redundant.

Theorem 4.4. *Under assumptions* **A1** *and* **A2′**, *for any* $D \geq 0$, *there exists (at least one) interior equilibrium state.*

Proof. The proof is an evident extension of that of Theorem 4.2 in [20]. ■

In our future research, we will extend the obtained results to the case of non-differentiable demand functions. However, some of the necessary techniques must be developed now in the differentiable case.

4.2. Case of Duopoly

For the duopoly case, we denote the value of the demand function's derivative by $\tau = G'(p)$ and rewrite the consistency Eqs. (27) and (28) as follows:

$$v_0 = \frac{1}{\displaystyle\frac{1}{a_1 + v_1} - \tau}, \quad \ldots \ldots \ldots \ldots \ldots \quad (30)$$

and

$$v_1 = \frac{1}{\displaystyle\frac{1}{a_0 + (1-\beta)v_0} - \tau}, \quad \ldots \ldots \ldots \quad (31)$$

where $\tau = [-\infty, 0]$. If $\tau = -\infty$ then system (30)–(31) has the unique solution $v_i(\tau) = 0$, $i = 0, 1$. The following result was demonstrated in [21].

Theorem 4.5. [21] *Under assumptions* **A1–A3** *and for any* $\tau \in (-\infty, 0]$ *there exists a unique solution of Eqs.* (30) *and* (31), *and this one-to-one correspondence is a continuous function of the variable* τ. *Moreover,* $v_i(\tau) \to 0$ *when* $\tau \to -\infty$, *and* $v_i(\tau)$ *strictly increases up to* $v_i(0)$ *as* τ *increases and tends to zero,* $i = 0, 1$. ■

The analogous result for the economies of scale model the analogous result is shown below.

Theorem 4.6. *Under assumptions* **A1** *and* **A2′** *and for any* $\tau \in (-\infty, 0]$, *there exists a unique solution of Eqs.* (30) *and* (31), *and this one-to-one correspondence is a continuous function of the variable* τ. *Moreover,* $v_i(\tau) \to 0$ *when* $\tau \to -\infty$, $v_1(\tau)$ *strictly increases up to* $v_1(0)$, *and* $v_0(\tau)$ *strictly increases up to* $\min\{v_i(0), \bar{u}_0\}$ *as* τ *grows and tends to zero,* $i = 0, 1$. *Here,* $\bar{u}_0$ *is as defined by* (14).

Proof. The proof of this theorem is quite long and will be published elsewhere. ■

Remark 4.4. The only difference of Theorem 4.6 from the corresponding existence result for the interior equilibrium in an economies-of-scale mixed oligopoly model is that in the latter, assumption **A2** is replaced with assumption **A2′**, while assumption **A3** becomes redundant. ■

Theorem 4.7. *Under assumptions* **A1** *and* **A2′**, *for any* $D \geq 0$, *there exists (at least one) interior equilibrium state.*

Proof. The proof is an evident extension of that of Theorem 4.2 in [20]. ■

4.3. Case of Oligopoly

Again, as in Subsection 4.2, denote the value of the demand function's derivative by $\tau = G'(p)$ and rewrite the consistency Eqs. (27) and (28) as follows:

$$v_0 = \frac{1}{\displaystyle\sum_{i=1}^{n} \frac{1}{a_i + v_i} - \tau}, \quad \ldots \ldots \ldots \ldots \quad (32)$$

and

$$v_i = \frac{1}{\dfrac{a_0+v_0}{a_0+(1-\beta)v_0}\displaystyle\sum_{j\neq i, j=1}^{n}\frac{1}{a_j+v_j}-\tau}, \quad . \quad . \quad . \quad (33)$$

$i=1,\ldots,n$, where $\tau=[-\infty,0]$. If $\tau=-\infty$, system (32) and (33) has the unique solution $v_i(\tau)=0,\ i=0,1,\ldots,n$.

The following result was demonstrated in [22].

Theorem 4.8. [22] *Under assumptions* **A1–A3** *and for any* $\tau\in(-\infty,0]$*, there exists a unique solution of Eqs.* (32) *and* (33) *that continuously depends on* τ*. Moreover,* $v_i(\tau)\to 0$ *when* $\tau\to-\infty$*,* $i=0,1,\ldots,n$*, and* $v_0(\tau)$ *strictly increases up to* $\min\{v_0(0),\bar{v}_0\}$ *as* τ *increases and tends to zero, if*

$$\frac{ns}{ns+a_0(n-1)^2}<\beta\le 1, \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ (34)$$

where $s=\max\{\bar{v}_0,a_0,a_1,\ldots,a_n\}$. ■

The analogous result for the economies of scale model is shown below.

Theorem 4.9. *Under assumptions* **A1** *and* **A2′** *and for any* $\tau\in(-\infty,0]$ *there exists a unique solution of Eqs.* (32) *and* (33)*, and this one-to-one correspondence is a continuous function of the variable* τ*. Moreover,* $v_i(\tau)\to 0$ *when* $\tau\to-\infty$*,* $v_i(\tau)$ *strictly increases up to* $v_i(0)$*,* $i=1,\ldots,n$*, and* $v_0(\tau)$ *strictly increases up to* $\min\{v_0(0),\bar{u}_0\}$ *as* τ *grows and tends to zero if*

$$\frac{ns}{ns+a_0(n-1)^2}<\beta\le 1, \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ (35)$$

where $s=\max\{\bar{u}_0,a_0,a_1,\ldots,a_n\}$*. Here,* $\bar{u}_0$ *is determined by expression* (13).

Proof. The proof of this theorem is extremely long and will be published elsewhere. ■

5. A Linear Demand Function

Again, following [21] and [22], we consider a particular case of the linear demand function by introducing a new assumption in place of **A1**.

A4. The demand function is linear, $G(p)=-Kp+T$ with $K>0, T>0$, and the ratio $T/K>0$ is large enough to ensure that $G(p)>0$ for all equilibrium states that can occur in the model.

5.1. A Linear Demand Function in Duopoly

Several interesting results concerning the behavior of the interior and exterior equilibria and their dependence on the parameter $\beta\in(0,1]$ have been derived and published in [21].

Theorem 5.1. [21] *For each* $\beta\in(0,1]$*, under assumptions* **A2–A4***, there exists uniquely an interior equilibrium state* $(p^*,q_0^*,q_1^*,v_0^*,v_1^*)=$ $=(p^*(\beta),q_0^*(\beta),q_1^*(\beta),v_0^*(\beta),v_1^*(\beta))$*. Moreover, the consistent coefficients of influence* $v_i^*=v_i^*(\beta)$*,* $i=0,1$*, the equilibrium production volumes* $q_i^*=q_i^*(\beta)$*,* $i=0,1$*, and the equilibrium price* $p^*=p^*(\beta)$*, which are treated as (well-defined) functions of the variable* β*, are continuously differentiable over the feasible domain* $\beta\in(0,1]$. ■

It is straightforward that the parameter β can be interpreted as a measure of the "nationalization" of company $i=0$ (*cf.* [11]). Indeed, the smaller the value of β, the higher the relative weight of the net profit in the company's objective function (4). In contrast, when $\beta\to 1$, the public company $i=0$ tends to behave more and more like the player maximizing the domestic social surplus. Therefore, it is intuitively clear that when the parameter β grows, the output produced by firm $i=0$ must increase, whereas the private company $i=1$ should decrease its supply because of the decreasing price. Furthermore, it is also intuitively clear that when β increases, the total equilibrium demand $G^*=G(p^*(\beta))$ must increase, thus decreasing the clearing (equilibrium) price $p^*(\beta)$. This leads both agents (private and public) in the market to a loss in their influence rates, that is, a decrease in their influence coefficients $v_i^*(\beta)$, $i=0,1$. For the particular case of the linear demand function, all these results can be demonstrated by mathematically rigorous arguments, as illustrated by the following result from [21].

Theorem 5.2. [21] *Under assumptions* **A2–A4***, the consistent coefficients of influence* $v_i^*(\beta)$*,* $i=0,1$ *and the equilibrium price* $p^*(\beta)$*, which are treated as (well-defined) functions of the variable* $\beta\in(0,1]$*, strictly decrease over the whole interval* $(0,1]$*. In addition, there exists a value* $\bar{\beta}\in[0,1)$ *such that the interior equilibrium private volume function* $q_1^*=q_1^*(\beta)$ *strictly decreases, while the interior equilibrium public volume function* $q_0^*=q_0^*(\beta)$ *strictly increases over the (semi-closed) interval* $\beta\in(\bar{\beta},1]$. ■

Remark 5.1. The obtained results lead to the conclusion that when a certain "degree of nationalization" $\bar{\beta}\in[0,1)$ is achieved, the private company is "crestfallen" and drops in both its production volume q_1 and its self-evaluation parameter v_1. However, for the consumers, the growing of β is the good news since the total production volume increases, whereas the clearing price p, vice versa, falls down. ■

Remark 5.2. As in the previous section, it is noteworthy that to obtain the corresponding results for the economies-of-scale model, it is sufficient to replace assumption **A2** with assumption **A2′** and to delete assumption **A3**, which is unnecessary. Therefore, the following theorems are true (the proofs are very long and will be published elsewhere). ■

Theorem 5.3. *For each* $\beta\in(0,1]$*, under assumptions* **A2′** *and* **A4***, there exists uniquely an interior equilibrium state* $(p^*,q_0^*,q_1^*,v_0^*,v_1^*)=$ $=(p^*(\beta),q_0^*(\beta),q_1^*(\beta),v_0^*(\beta),v_1^*(\beta))$*. Moreover, the consistent coefficients of influence* $v_i^*=v_i^*(\beta)$*,* $i=0,1$*, the equilibrium production volumes* $q_i^*=q_i^*(\beta)$*,* $i=0,1$*, and the equilibrium price* $p^*=p^*(\beta)$*, which are treated as (well-defined) functions of the variable* β*,* *are continuously differentiable over the feasible domain* $\beta \in (0,1]$. ■

Theorem 5.4. *Under assumptions* **A2′** *and* **A4**, *the consistent coefficients of influence* $v_i^*(\beta)$, $i = 0,1$ *and the equilibrium price* $p^*(\beta)$, *which are treated as (well-defined) functions of the variable* $\beta \in (0,1]$, *strictly decrease over the whole interval* $(0,1]$. *In addition, there exists a value* $\bar{\beta} \in [0,1)$ *such that the interior equilibrium private volume function* $q_1^* = q_1^*(\beta)$ *strictly decreases, while the interior equilibrium public volume function* $q_0^* = q_0^*(\beta)$ *strictly increases over the (semi-closed) interval* $\beta \in (\bar{\beta},1]$. ■

5.2. A Linear Demand Function in an Oligopoly

For the general oligopoly ($n > 1$), several interesting results concerning the behavior of the interior and exterior equilibria dependent on the parameter $\beta \in (0,1]$ were derived and published in [22].

Theorem 5.5. [22] *For each* $\beta \in (0,1]$, *under assumptions* **A2–A4**, *there exists uniquely an interior equilibrium state* $(p^*,q_0^*,q_1^*,\ldots,q_n^*,v_0^*,v_1^*,\ldots,v_n^*)= =(p^*(\beta),q_0^*(\beta),q_1^*(\beta),\ldots,q_n^*(\beta),v_0^*(\beta),v_1^*(\beta),\ldots,v_n^*(\beta))$. *Moreover, the consistent coefficients of influence* $v_i^* = v_i^*(\beta)$, $i = 0,1,\ldots,n$, *which are treated as (well-defined) functions of the variable* β, *are continuously differentiable over the feasible domain*

$$\frac{ns}{ns + a_0(n-1)^2} < \beta \le 1, \quad \ldots\ldots\ldots \quad (36)$$

where $s = \max\{\bar{v}_0, a_0, a_1, \ldots, a_n\}$. ■

It is straightforward that the parameter β can be interpreted as a measure of the "nationalization" of company $i = 0$ (*cf.* [11]). Indeed, the smaller the value of β, the higher the relative weight of the net profit in the company's objective function (4). In contrast, when $\beta \to 1$, the public company $i = 0$ tends to behave increasingly like the player maximizing the domestic social surplus. Therefore, it is intuitively clear that when the parameter β grows, the output produced by firm $i = 0$ must increase, whereas the private companies $i = 1,\ldots,n$ should decrease their supply because of the decreasing price. Furthermore, it is also intuitive that when β increases, the total (passive) demand $G^* = G(p^*(\beta))$ must increase, thus dropping the clearing (equilibrium) price $p^*(\beta)$. The latter evidently leads all the agents (private and public) of the market to losses in their influence rates,that is, to decreases in their influence coefficients $v_i^*(\beta)$, $i = 0,1,\ldots,n$. All these properties are illustrated in the next section for the particular case of the linear demand function via the results of numerical experiments.

Remark 5.3. The obtained results lead to the conclusion that when a certain "degree of socialization" $\bar{\beta} \in (0,1)$ achieved, the private companies are "crestfallen" and drop both their production volumes q_i and their self-evaluation parameters v_i . However, the increase of β is good for consumers since the total production volume increases, whereas the clearing price p, vice versa, goes down. ■

Remark 5.4. The threshold $\bar{\beta} \in (0,1)$ need not be tending to zero, as shown via the numerical experiments with the linear demand functions described in the next section. ■

Remark 5.5. As in the previous section, it is noteworthy that to obtain the corresponding results for the case of the economies-of-scale model, it is enough to replace assumption **A2** with assumption **A2′** and to delete assumption **A3**, which is redundant. Therefore, the following theorems are true (the proofs are quite long and will be published elsewhere). ■

Theorem 5.6. *For each* $\beta \in (0,1]$, *under assumptions* **A2′** *and* **A4**, *there exists uniquely an interior equilibrium state* $(p^*,q_0^*,q_1^*,\ldots,q_n^*,v_0^*,v_1^*,\ldots,v_n^*)= =(p^*(\beta),q_0^*(\beta),q_1^*(\beta),\ldots,q_n^*(\beta),v_0^*(\beta),v_1^*(\beta),\ldots,v_n^*(\beta))$. *Moreover, the consistent coefficients of influence* $v_i^* = v_i^*(\beta)$, $i = 0,1,\ldots,n$, *which are treated as (well-defined) functions of the variable* β, *are continuously differentiable over the feasible domain*

$$\frac{ns}{ns + a_0(n-1)^2} < \beta \le 1, \quad \ldots\ldots\ldots \quad (37)$$

where $s = \max\{\bar{u}_0, a_0, a_1, \ldots, a_n\}$. ■

6. Numerical Experiments: Oligopoly

To illustrate the difference between the mixed oligopoly studied in this paper and the standard mixed and the classical oligopoly cases related to the CVE with consistent conjectures (influence coefficients), in our previous paper [22], we applied Eqs. (27) and (28) to a simple example of oligopoly in the electricity market from [20] and [23]. The only difference in our modified example from [23] is that in their case, all six agents (suppliers) are private companies producing electricity and maximizing their net profits, while in our example examined in [22], similar to [20], we assume that agent 0 (agent 2 in some instances) is a public enterprise seeking to maximize the convex combination of the domestic social surplus and its profit described in (4), while the other generators are private firms maximizing their net profits. Similar numerical experiments were conducted and reported in [20], but only for $\beta = 1$. All other parameters involved in the inverse demand function $p = p(G,D)$ and the producers' cost functions are exactly the same as in [20].

Following the above-mentioned references, we selected the IEEE six-generator, 30-bus system (*cf.* [23]) to illustrate our analysis. The inverse demand function in the electricity market has the form:

$$\begin{aligned} p(G,D) &= 50 - 0.02(G+D) \\ &= 50 - 0.02(q_0 + q_1). \quad \ldots\ldots\ldots \quad (38) \end{aligned}$$

Here, agents $i = 0,1,\ldots,5$ will be combined in the various examples listed below. In particular, Oligopoly 1 will involve agents $i = 0$ (public) and $j = 1,\ldots,5$ (private), whereas Oligopoly 2 comprises agents $i = 5$ (public) and $j = 0,1,\ldots,4$ (private).

To find the consistent influence coefficients in the clas-

Table 1. Cost function parameters.

Agent i	b_i	a_i
0	2.00	0.02000
1	1.75	0.01750
2	3.00	0.02500
3	3.00	0.02500
4	1.00	0.06250
5	3.25	0.00834

Table 2. Coefficients of influence w_i for Oligopoly 1 (C.: – Cournot, P.: – Perfect Competition).

i	$\beta = 0$	$1/4$	$1/2$	$3/4$	$\beta = 1$	C.	P.
0	0.193	0.192	0.190	0.189	0.188	1.0	0.0
1	0.196	0.189	0.182	0.174	0.166	1.0	0.0
2	0.188	0.180	0.173	0.166	0.159	1.0	0.0
3	0.188	0.180	0.173	0.166	0.159	1.0	0.0
4	0.178	0.168	0.161	0.154	0.148	1.0	0.0
5	0.224	0.216	0.208	0.201	0.193	1.0	0.0

Table 3. Consistent equilibrium (production volumes q_i, total volume G, price p, and the objective functions' values) for Oligopoly 1.

	$\beta = 0$	$\beta = 1/4$	$\beta = 1/2$	$\beta = 3/4$	$\beta = 1$
q_0	353.405	421.555	489.706	557.856	626.006
q_1	405.120	393.375	381.629	369.883	358.138
q_2	258.436	248.940	239.444	229.947	220.451
q_3	258.436	248.940	239.444	229.947	220.451
q_4	142.898	138.539	134.180	129.821	125.462
q_5	560.180	542.361	524.543	506.723	488.905
G	1,978.5	1,993.7	2,008.9	2,031.8	2,039.4
p	10.43	10.125	9.82	9.515	9.21
S	1,727.4	11,842.5	21,957.6	32,072.7	42,187.8
π_1	2,076.6	1,944.96	1,813.32	1,681.68	1,550.04
π_2	1,082.9	1,002.65	922.4	842.15	761.90
π_3	1,082.9	1,002.65	922.4	842.15	761.90
π_4	707.48	665.20	622.93	580.65	538.37
π_5	2,709.8	2,511.85	2,313.89	2,115.93	1,917.98

Table 4. Cournot equilibrium (production volumes q_i, total volume G, price p, and the objective functions' values) for Oligopoly 1.

	$\beta = 0$	$\beta = 1/4$	$\beta = 1/2$	$\beta = 3/4$	$\beta = 1$
q_0	319.06	539.295	759.53	979.765	1,200.00
q_1	347.00	312.15	277.30	242.45	207.60
q_2	261.39	232.35	203.31	174.26	145.22
q_3	261.39	232.35	203.31	174.26	145.22
q_4	166.82	150.98	135.14	119.29	103.45
q_5	406.23	360.11	314.00	267.88	221.77
G	1,761.9	1,827.2	1,892.6	1,990.6	2,023.3
p	14.76	13.45	12.15	10.84	9.53
S	3,054.0	11,184.9	19,315.8	27,446.6	35,577.5
π_1	3,461.7	2,906.03	2,350.36	1,794.69	1,239.02
π_2	2,220.5	1,836.72	1,452.94	1,069.16	685.38
π_3	2,220.5	1,836.72	1,452.94	1,069.16	685.38
π_4	1,426.2	1,206.78	987.36	767.93	548.51
π_5	3,988.5	3,288.55	2,588.6	1,888.65	1,188.70

Table 5. Perfect competition equilibrium (production volumes q_i, total volume G, price p, and the objective functions' values) for Oligopoly 1.

	$\beta = 0$	$\beta = 1/4$	$\beta = 1/2$	$\beta = 3/4$	$\beta = 1$
q_0	348.43	348.43	348.43	348.43	348.43
q_1	412.49	412.49	412.49	412.49	412.49
q_2	238.74	238.74	238.74	238.74	238.74
q_3	238.74	238.74	238.74	238.74	238.74
q_4	127.50	127.50	127.50	127.50	127.50
q_5	685.68	685.68	685.68	685.68	685.68
G	2,051.6	2,051.6	2,051.6	2,051.6	2,051.6
p	8.97	8.97	8.97	8.97	8.97
S	1,214.0	11,736.4	22,258.8	32,781.1	43,303.5
π_1	1,488.80	1,488.80	1,488.80	1,488.80	1,488.80
π_2	712.47	712.47	712.47	712.47	712.47
π_3	712.47	712.47	712.47	712.47	712.47
π_4	507.98	507.98	507.98	507.98	507.98
π_5	1,960.50	1,960.50	1,960.50	1,960.50	1,960.50

sical oligopoly market (Case 1, $\beta = 0$), [23] uses Eq. (25) for all six agents, while for the partially mixed oligopoly models (Oligopoly 1 or 2, $\beta > 0$), we exploit Eq. (23) for the public agent (which is agent 0 in Oligopoly 1 and agent 5 in Oligopoly 2) and Eq. (24) for the private companies (that is, 1 through 5 in Oligopoly 1 and 0 through 4 in Oligopoly 2), with $0 < \beta < 1$. Of course, when $\beta = 1$, our model coincides with the mixed oligopoly studied in [18]. With the thus-obtained influence coefficients, the (unique) equilibrium is found for Oligopoly 1 and 2. The equilibrium results (influence coefficients, production outputs in MWh, equilibrium price, and the objective functions' optimal values in dollars per hour) are presented in **Tables 2–9**. To make our conjectures v_i comparable to those used in [20], [18], and [23], we divide them by $[-p'(G)] = K^{-1} = 0.02$ and thus obtain $w_i := -v_i/p'(G) = Kv_i = 50v_i$, $i = 0, 1, \ldots, n$, as shown in **Tables 2** and **6**, where the *Cournot* and *Perfect* columns show the influence coefficients for the Cournot-Nash and perfect competition models, respectively.

Tables 3–5 from [22] show the numerical results for Oligopoly 1.

As **Table 3** clearly shows, the market clearing price (equilibrium price) in the case of the classical oligopoly ($\beta = 0$) is $p_1 = \$10.43$, which is higher than the mixed oligopoly equilibrium price $p_2 = \$9.21$. The assertions of Remark 5.3 are also well confirmed: the total production volume grows together with the public firm's output and the domestic social surplus, while the clearing price (as well as the private companies' outputs and net profits) decrease when β increases from 0 to 1. The conclusion can be made — that the higher the proportion of domestic social surplus in the public firm's objective, the greater the total production volume, and hence, the lower the clearing price of electricity.

Table 6. Coefficients of influence w_i for Oligopoly 2 (C.: Cournot, P.: Perfect Competition).

i	$\beta=0$	1/4	1/2	3/4	$\beta=1$	C.	P.
0	0.193	0.178	0.162	0.147	0.132	1.0	0.0
1	0.196	0.181	0.166	0.150	0.135	1.0	0.0
2	0.188	0.173	0.158	0.143	0.128	1.0	0.0
3	0.188	0.173	0.158	0.143	0.128	1.0	0.0
4	0.175	0.161	0.147	0.133	0.118	1.0	0.0
5	0.224	0.223	0.220	0.218	0.216	1.0	0.0

Table 7. Consistent equilibrium (production volumes q_i, total volume G, price p, and the objective functions' values) for Oligopoly 2.

	$\beta=0$	$\beta=1/4$	$\beta=1/2$	$\beta=3/4$	$\beta=1$
q_0	353.405	329.924	306.443	282.961	259.480
q_1	405.120	379.647	354.175	328.702	303.229
q_2	258.436	238.048	217.660	197.272	176.884
q_3	258.436	238.048	217.660	197.272	176.884
q_4	142.898	133.670	124.441	115.213	105.984
q_5	560.180	691.081	821.983	952.884	1,083.785
G	1,978.5	2,010.4	2,042.4	2,090.3	2,106.3
p	10.43	9.79	9.15	8.51	7.88
π_0	1,727.40	1,508.34	1,289.28	1,070.22	851.16
π_1	2,076.6	1,820.64	1,564.68	1,308.71	1,052.75
π_2	1,082.9	929.98	777.06	624.14	471.22
π_3	1,082.9	929.98	777.06	624.14	471.22
π_4	707.48	625.02	542.56	460.09	377.63
S	2,709.8	13,151.7	23,593.6	34,035.4	44,477.3

It is also interesting to compare the results in the CVE with consistent conjectures against the production volumes and profits obtained for the same cases in the classical Cournot equilibrium (i.e., with all $w_i = 1$, $i = 0,1,\ldots,5$). **Table 4** provides the numerical results: $p_3 =$ \$14.76 in the classical oligopoly ($\beta = 0$), which is much higher than the market equilibrium price, $p_4 =$ \$9.535, in the mixed oligopoly ($\beta = 1$), which is only 65% of the former.

Again, the total electricity production level is monotone increasing along with parameter β, starting from $G_3 = 1761.90$ MWh when $\beta = 0$ and ending with $G_4 = 2023.256$ MWh for $\beta = 1$. Another interesting observation can be made by comparing **Tables 3** and **4**: when β is small or medium ($\beta \leq 0.75$), strong private companies (such as agent 5) have higher objective function values by making use of the Cournot conjectures $w_i = 1$, $i = 0,\ldots,5$. However, if β is high enough (i.e., greater than 0.75), the orderings are reversed: by relying on the Bulavsky consistent conjectures calculated by Eqs. (27) and (28) instead of the Cournot-Nash conjectures, the private companies improve their results significantly.

We also consider the perfect competition model (see **Table 5**) with $w_i = 0$, $i = 0,\ldots,5$, which naturally gives the same results for all values of β and which is the best for consumers. In our example, this model overcomes the mixed oligopoly with consistent conjectures, both in terms of the market price, $p_5 =$ \$8.97, and the total production volume $G_5 = 2051.57$ MWh. The domestic social surplus (with $\beta = 1$) is also slightly higher in this case (of perfect competition), \$43,303.52 per hour, than that in the mixed oligopoly with consistent conjectures (also $\beta = 1$), $\$42,187.80$ per hour.

Table 8. Cournot equilibrium (production volumes q_i, total volume G, price p, and the objective functions' values) for Oligopoly 2.

	$\beta=0$	$\beta=1/4$	$\beta=1/2$	$\beta=3/4$	$\beta=1$
q_0	319.06	269.95	220.84	171.72	122.61
q_1	347.00	294.61	242.23	189.84	137.45
q_2	261.39	217.73	174.08	130.42	86.77
q_3	261.39	217.73	174.08	130.42	86.77
q_4	166.82	143.01	119.19	95.38	71.57
q_5	406.23	717.08	1,027.92	1,338.77	1,649.61
G	1,761.9	1,860.1	1,958.3	2,105.7	2,154.8
p	14.76	12.80	10.83	8.87	6.90
π_0	3,054.0	2,403.25	1,752.51	1,101.76	451.01
π_1	3,461.7	2,732.07	2,002.44	1,272.81	543.18
π_2	2,220.5	1,726.54	1,232.59	738.63	244.67
π_3	2,220.5	1,726.54	1,232.59	738.63	244.67
π_4	1,426.2	1,135.28	844.36	553.43	262.51
S	3,988.5	13,269.3	22,550.0	31,830.8	41,111.6

Table 9. Perfect competition equilibrium (production volumes q_i, total volume G, price p, and the objective functions' values) for Oligopoly 2.

	$\beta=0$	$\beta=1/4$	$\beta=1/2$	$\beta=3/4$	$\beta=1$
q_0	348.43	348.43	348.43	348.43	348.43
q_1	412.49	412.49	412.49	412.49	412.49
q_2	238.74	238.74	238.74	238.74	238.74
q_3	238.74	238.74	238.74	238.74	238.74
q_4	127.50	127.50	127.50	127.50	127.50
q_5	685.68	685.68	685.68	685.68	685.68
G	2,051.6	2,051.6	2,051.6	2,051.6	2,051.6
p	8.97	8.97	8.97	8.97	8.97
π_0	1,214.0	1,214.0	1,214.0	1,214.0	1,214.0
π_1	1,488.80	1,488.80	1,488.80	1,488.80	1,488.80
π_2	712.47	712.47	712.47	712.47	712.47
π_3	712.47	712.47	712.47	712.47	712.47
π_4	507.98	507.98	507.98	507.98	507.98
S	1,960.50	12,482.9	23,005.27	33,527.7	44,050.0

Next, we numerically estimate Oligopoly 2, where private companies 0 – 4 compete with a much stronger public company 5 (see **Table 1** for the parameters). The consistent coefficients of influence computed by Eqs. (27) and (28) are shown in **Table 6**.

Tables 7–9 show the numerical results for Oligopoly 2.

For the numerical results for Oligopoly 2, similar comments may be formulated to those for Oligopoly 1. For instance, as **Table 7** apparently shows, the market clearing price (equilibrium price) in the case of the classical oligopoly ($\beta = 0$) is quite elevated, reaching $p_6 =$ \$20.60, in comparison to the mixed oligopoly equilibrium price, $p_7 =$ \$15.85, which is 25% lower. The modes of behavior predicted by Theorems 5.5 and 5.6 and by Remark 5.3

are also confirmed: the total production volume increases together with the public firm's output and the domestic social surplus, while the clearing price (as well as the private company's output and net profit) decreases when β grows from 0 to 1. As for Oligopoly 1, the higher the proportion of the domestic social surplus in the public firm's objective function, the greater the total production volume, and hence, the lower the clearing price of electricity.

Again, it is worthwhile to compare the results in the CVE with consistent conjectures against the production volumes and profits obtained for the same cases in the classical Cournot equilibrium (i.e., with all $w_i = 1$, $i = 0,1$). **Table 8** presents the numerical results, where $p_8 =$ \$24.16 in the classical oligopoly ($\beta = 0$) is substantially greater than the market equilibrium price $p_9 =$ \$16.59 in the mixed oligopoly ($\beta = 1$).

Similar to Oligopoly 1, the total electricity production level is monotone increasing along with parameter β, starting from $G_6 = 1470.23$ MWh when $\beta = 0$ and ending with $G_7 = 1707.68$ MWh for $\beta = 1$. An analogous feature can be found by comparing **Tables 7** and **8**: when β is small or medium ($\beta \leq 0.75$), the private companies have higher objective function values by making use of the Cournot conjectures $w_i = 1, i = 0,\dots,4$. However, for β greater than 0.75, the orderings are reversed: by relying on the Bulavsky consistent conjectures calculated by Eqs. (27) and (28) instead of the Cournot-Nash conjectures, the strong private firms ($i = 0$ and $i = 1$) improve their profits significantly.

We also consider the perfect competition model (see **Table 9**) with $w_i = 0$, $i = 0,\dots,5$, which naturally gives the same results for all values of β and which is known to be best for consumers. Indeed, in contrast to Oligopoly 1, in Oligopoly 2, the perfect competition results are superior (from the consumers' point of view) to those of the mixed oligopoly with consistent conjectures, both in terms of the market clearing price, $p_{10} =$ \$13.60, and the total production volume, $G_8 = 1820.23$ MWh. In line with this, the domestic social surplus (with $\beta = 1$) is considerably higher in this case (that of perfect competition), \$36,493.68 per hour, than that in the mixed oligopoly with consistent conjectures (also $\beta = 1$), which is \$32,875.44 per hour.

Finally, by comparing pairwise **Tables 3** and **7** with **Tables 4** and **8**, we can see that the latter tables contain higher total production volumes and lower clearing prices than the former. These results may serve as a good example of how a strong private company may implicitly regulate the market price within a (mixed) oligopoly: the stronger the private company, the better the results for consumers.

Remark 6.1. Comparing pairwise **Tables 3–4** with **Tables 7–8**, an interesting phenomenon is apparent. If all companies are private ($\beta = 0$), the Cournot-Nash equilibrium is more attractive for them than the Bulavsky equilibrium (the consistent conjectural variations equilibrium) since the former provides higher profits to companies than the latter. However, the relationship is exactly opposite (at least, for strong private agents) when company $i = 0$ (or $i = 5$) is public. In this case, the Bulavsky equilibrium is more profitable for the private agents than the Cournot-Nash equilibrium. At the same time, the energy price is always lower in the Bulavsky equilibrium than in the Cournot-Nash equilibrium. In other words, if the energy market is not (implicitly) regulated by the intervention of a public company that strives to enhance the domestic social surplus, the Cournot-Nash equilibrium proves to be better for private suppliers, while the Bulavsky equilibrium is more attractive for consumers because of the lower clearing price. On the other hand, when a public company operates in the market, the Bulavsky equilibrium is better for both suppliers and consumers than the Cournot-Nash equilibrium.
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Remark 6.2. Reconsider the phenomenon described in Remark 6.1 and suppose that a municipality (government) of a location consuming energy from the market with any value of $\beta \in [0,1]$ is responsible enough to issue subsidies to either (*i*) reduce the Cournot-Nash price for the consumer to the price proposed by the Bulavsky equilibrium or (*ii*) compensate the private suppliers' profit losses whenever they switch to Bulavsky conjectures instead of Cournot-Nash ones. Simple calculations demonstrate that option (*ii*) is *always* cheaper for the location's administration than option (*i*), that is, if they pay subsidies to producers they save a great deal compared to paying them (i.e., subsidies) to consumers. Moreover, according to Theorems 5.5 and 5.6, it is possible to show that there exists a (unique) value of the convex combination coefficient $\beta^* \in (0,1)$ such that if the "public" company ($i = 0$) weights the domestic social surplus part of its utility function (4) by the parameter β^*, the private companies' aggregate profits in the Cournot-Nash equilibrium and in the Bulavsky equilibrium will be equal. This implies that when the "nationalization degree" of firm $i = 0$ corresponds to the optimal percentage value $\beta = \beta^*$, there is no need to pay subsidies either to private suppliers (because they are indifferent as to what conjectures to apply, the Bulavsky or Cournot-Nash ones) or to consumers (since all suppliers are inclined to produce Bulavsky equilibrium outputs, which always generates a lower price for consumers than the Cournot-Nash price). In other words, the aforementioned value of $\beta = \beta^*$ could be interpreted as a kind of "optimal" percentage of state-owned shares in the "public" company's assets. ■

7. Conclusion

In this paper, we considered a model of a combined mixed oligopoly with the conjectural variations equilibrium (CVE). The agents' conjectures concern the price variations that depend on the increase or decrease of their production outputs. We established the existence and uniqueness results for the CVE (called the *exterior equilibrium*) for any set of feasible conjectures. To introduce the notion of the *interior equilibrium*, we developed a consistency criterion for the conjectures (referred to as influence coefficients) and proved the existence theorem for the interior equilibrium (understood as the CVE with consistent conjectures). The special cases of the models with economies of scale and markets with affine (linear) demand functions were also examined.

To prepare the base for the extension of our results to the case of non-differentiable demand functions, we also investigated the behavior of consistent conjectures that depend on a parameter representing the demand function's derivative with respect to the market price. Numerical experiments with a small electrical power market have been conducted, and some comparisons of the consistent CVE (Bulavsky's conjectures) against the classical Cournot-Nash and perfect competition equilibria have been made.

In our forthcoming papers, we will examine the qualitative behavior of prices and production outputs when the demand function is not necessarily differentiable and the cost functions are not quadratic. Moreover, the results outlined in Remarks 6.1 and 6.2 will receive a mathematically rigorous justification.

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