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Paper:

Analysis of Consistent Equilibria in a Mixed Duopoly

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This paper examines a model of a mixed duopoly with conjectural variations equilibrium (CVE), in which one of the agents maximizes a convex combination of his/her net profit and domestic social surplus. The agents' conjectures concern the price variations, which depend on their production output variations. Based on the already established existence and uniqueness results for the CVE (called the *exterior equilibrium*) for any set of feasible conjectures, the notion of interior equilibrium is introduced by developing a consistency criterion for the conjectures (referred to as influence coefficients), and the existence theorem for the interior equilibrium (understood as a CVE state with *consistent conjectures*) is proven. When the convex combination coefficient tends to 1, thus transforming the model into the mixed duopoly in its extreme form, two trends are apparent. First, for the private company, the equilibrium with consistent conjectures becomes more proficient than the Cournot-Nash equilibrium. Second, there exists a (unique) value of the combination coefficient such that the private agent's profit is the same in both of the above-mentioned equilibria, which makes subsidies to the producer or to consumers unnecessary.

Keywords: management engineering, game theory, equilibrium theory

1. Introduction

Recently, models of mixed oligopolies have become very popular in the literature. In contrast to a classical oligopoly, a mixed oligopoly usually includes at least one special agent in addition to the standard agents who maximize their net profits. A special company may deal with an objective function that is distinct from the net profit. Many such models include an agent who maximizes the domestic social surplus (*cf.* [1–5]). An income-per-worker function replaces the standard net profit objective function in some other papers (*cf.* [6–9]). Researchers [10] and [11] examine a third kind of mixed duopoly in which an exclusive participant aims to enhance a convex combination of his/her net profit and the domestic social surplus.

Almost all the above-mentioned works study the mixed oligopoly using the classical Cournot, Hotelling, or Stackelberg models. The concept of *conjectural variations equilibrium* (CVE) introduced by Bowley [12] and Frisch [13] is another possible solution form, and its use has increased recently. In the CVE, players behave as follows: each agent chooses her/his most favorable strategy, having supposed that every opponent's action is a *conjectural variation function* of her/his own move. For example, as Laitner [14, p. 643] states, "Although the firms make their output decisions simultaneously, plan changes are always possible before production begins." In other words, in contrast to the Cournot–Nash approach, here, every firm supposes that its choice of output level will affect its rivals' behavior. Thus, the anticipation (or conjectural variation) function that arises is the core of conjectural variation decision making, or the CVE.

As stated in [15] and [16], the concept of the CVE has been the topic of numerous theoretical disputes (*cf.* [17]). Nevertheless, economists have made extensive use of various forms of the CVE to predict the outcome of noncooperative behavior in many areas of economics. The literature on conjectural variations has focused mainly on two-player games (*cf.* [15]) because a serious conceptual difficulty arises if the number of agents is greater than two (*cf.* [15] and [18]).

In order to overcome this conceptual hurdle, which arises in many-player games, Bulavsky, in [19], introduced a new approach. Rather than assuming the equivalence (symmetry) of players in the oligopoly, it is supposed that every player does *not* make conjectures redarding the (optimal) response functions of each of the other players, but only regarding variations in the market clearing price, which depend on (infinitesimal) variations in the same agent's output volume. Knowing the opponents' conjectures (the *influence coefficients*), every agent applies a verification procedure and reveals whether her/his influence coefficient is *consistent* with those of the rest of the players.

In the recent papers [18] and [20], the results of [19] were extended to the mixed duopoly and oligopoly cases, respectively. In both papers, the *exterior equilibrium* was defined as a CVE state with influence coefficients fixed in an exogenous mode. The existence and uniqueness theorems for this sort of CVE were established, to be used as a cornerstone of the concept of *interior equilibrium*, which is the exterior equilibrium with consistent conjectures (influence coefficients). The consistency criteria, consistency verification procedures, and existence theorems for the interior equilibrium were also formulated and proved in [18] and [20].

Recently, in [21], the above-described theoretical results were extended to the case of a partially mixed duopoly — that is, a model in which, similar to [10] and [11], the public company maximizes a convex combination of the net profit and the domestic social surplus. The results of numerical experiments with a test model of an electricity market from [22], both with and without a public company among the agents, showed the importance of the CVE for the consumer. In this paper, we explore this in greater detail, and an interesting result will be specified. When the convex combination coefficient tends to 1 — thus pushing the model toward the mixed duopoly in its extreme form — two trends appear. First, for the private company, the equilibrium with consistent conjectures becomes more proficient than that of the Cournot-Nash equilibrium. Second, there exists a (unique) value of the combination coefficient such that the private agent's profit is the same at both the above-mentioned equilibria, which makes subsidies from the authoritites to the producer or to consumers unnecessary.

The rest of the paper is organized as follows. Section 2 formulates the model and the two kinds of equilibrium we consider (exterior and interior). In Section 3, we present the main theorem showing the existence and uniqueness of the exterior equilibrium for any set of feasible conjectures (influence coefficients), as well as the formulas for the derivative of the equilibrium price p with respect to the active demand variable D. Section 4 deals with the consistency criterion and the definition of the interior equilibrium (which can be treated as a consistent CVE state, or CCVE); the CCVE existence theorem from [21] is also discussed. To provide tools for future research concerning the interrelationships between the demand structure (with a demand function that is not necessarily smooth) and the CVEs with consistent conjectures (influence coefficients), the behavior of the latter as functions of a certain parameter governed by the derivative of the demand function with respect to p is considered in Theorem 4.2 at the end of Section 4. In Section 5, a qualitative analysis of the results of the numerical experiments from [21] is presented, while concluding remarks are given in Section 6.

2. Mixed Duopoly with Combined Payoff Functions

Following [21], consider two suppliers of a homogeneous commodity with the cost functions $f_i(q_i)$, $i = 0,1$, where $q_i \geq 0$ is the output of supplier i. Consumer demand is described by a demand function $G(p)$, where p is the market price proposed by producers. The value of an active demand D is nonnegative and does not depend on the price. Any equilibrium in the model must determine the relationship between the demand and supply for a given price p provided by the following balance equality

$$q_0 + q_1 = G(p) + D. \qquad (1)$$

We also assume the following properties for the model's data.

A1. The demand function $G = G(p) \geq 0$ defined over $p \in [0, +\infty)$ is non-increasing and continuously differentiable.

A2. For both agents, $i = 0,1$, their cost functions $f_i = f_i(q_i)$ are quadratic:

$$f_i(q_i) = \frac{1}{2} a_i q_i^2 + b_i q_i, \qquad (2)$$

where $a_i > 0,\ b_i > 0,\ i = 0,1$. In addition, we suppose that

$$b_0 \leq b_1. \qquad (3)$$

Remark 2.1. Although the assumption of $a_i > 0$, $i = 0,1$, may appear to be unacceptable in view of the scale effects often observed in real-life production economies, it is not uncommon in the theories of classical and mixed oligopolies (*cf.* [3–5] and [22]). In the majority of cases, this assumption is the easiest way to provide for the concavity of each player's payoff function. However, this condition can be somewhat relaxed — like, for example, in [23], where the second derivative of the cost function is not assumed to be positive. Then the desired payoff function's concavity is achieved by another assumption that combines the first derivative of the demand function and the second derivative of the cost function. Finally, the scale effect can also be modeled by permitting the first-order coefficients b_i, $i = 0,1$, to be negative. We have already obtained the corresponding results for this more general case; they will be published elsewhere. ■

The private producer $i = 1$ chooses its output volume $q_1 \geq 0$ so as to maximize its profit function $\pi_1(p, q_1) = p \cdot q_1 - f_1(q_1)$. On the other hand, the public company with index $i = 0$ produces $q_0 \geq 0$ so as to maximize a convex combination of the domestic social surplus (defined as the difference between the consumer surplus, the private company's total revenue, and the public firm's production costs) and its net profit:

$$S(p; q_0, q_1) = \beta \left[\int_0^{q_0+q_1} p(x)dx - pq_1 - b_0 q_0 - \frac{1}{2} a_0 q_0^2 \right] + (1-\beta)\left(pq_0 - b_0 q_0 - \frac{1}{2} a_0 q_0^2 \right), \quad . \quad (4)$$

where $0 < \beta \leq 1$ (here, we follow [10] and [11]). We postulate that the agents (both public and private) assume that their choice of production volumes may affect the price value p. The latter assumption can be defined by a conjectured dependence of the price p on the output values q_i. Then, the first-order maximum condition to describe the equilibrium would have the following forms. For the public company ($i = 0$),

$$\frac{\partial S}{\partial q_0} = p - [\beta q_1 - (1-\beta) q_0] \frac{\partial p}{\partial q_0} - f_0'(q_0) \begin{cases} = 0, & \text{if } q_0 > 0; \\ \leq 0, & \text{if } q_0 = 0; \end{cases} \quad . \quad . \quad . \quad . \quad (5)$$

and for the private firm ($i = 1$),

$$\frac{\partial \pi_1}{\partial q_1} = p + q_1 \frac{\partial p}{\partial q_1} - f_1'(q_1) \begin{cases} = 0, & \text{if } q_1 > 0; \\ \leq 0, & \text{if } q_1 = 0. \end{cases} \quad . \quad . \quad . \quad . \quad (6)$$

Thus, we see that to describe the agent's behavior, we must evaluate the behavior of the derivative $\partial p / \partial q_i = -v_i$ rather than the exact functional dependence of p on q_i, $i = 0, 1$. Here, we introduce the negative sign (the minus) to deal with nonnegative values of v_i. Of course, the conjectured dependence of p on q_i must provide (at least locally) the concavity of the i-th agent's conjectured profit as a function of its output.

For instance, it suffices to assume the coefficient v_1 (henceforth referred to as the first agent's *influence coefficient*) to be nonnegative and constant. Then the conjectured local dependence of the private firm's profit variation on the production output's variation $(\eta_1 - q_1)$ has the form $[p - v_1(\eta_1 - q_1)]\, \eta_1 - pq_1 - f_1(\eta_1) + f_1(q_1)$, which implies that the profit is a concave function with respect to η_1. Therefore, the profit's maximum condition at $\eta_1 = q_1$ is provided by the relationships

$$\begin{cases} p = v_1 q_1 + b_1 + a_1 q_1, & \text{if } q_1 > 0; \\ p \leq b_1, & \text{if } q_1 = 0. \end{cases} \quad . \quad . \quad . \quad (7)$$

Similarly, the public company conjectures that the local dependence of the variation of its payoff function on its production output's variation $(\eta_0 - q_0)$ has the form

$$\beta \left\{ \int_{q_0+q_1}^{\eta_0+q_1} p(x)dx - [p - v_0(\eta_0 - q_0)]\, q_1 + pq_0 \right\} + (1-\beta)\left\{ [p - v_0(\eta_0 - q_0)]\, \eta_0 - pq_0 \right\} - f_0(\eta_0) + f_0(q_0), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

which implies that the public company's payoff function is concave with respect to η_0; thus, the maximum condition at $\eta_0 = q_0$ is written as follows:

$$\begin{cases} p = -\beta v_0 q_1 + (1-\beta) v_0 q_0 + b_0 + a_0 q_0, & \text{if } q_0 > 0; \\ p \leq -\beta v_0 q_1 + b_0, & \text{if } q_0 = 0. \end{cases} \quad (9)$$

Remark 2.2. If the agents' conjectures about the model were given exogenously, as assumed in [24] and [25], we would allow the values v_i to be functions of q_i and p. However, we use the approach from [18], [19], and [21], where the conjecture parameters for the equilibrium are determined simultaneously with price p and output values q_i by a special verification procedure. In the latter case, the influence coefficients are the scalar parameters determined only for the equilibrium. In Section 4, such an equilibrium state is called *interior* and is described by the set of variables and parameters (p, q_0, q_1, v_0, v_1). ■

3. Exterior Equilibrium in Duopoly

In order to present the verification procedure we need an intermediate notion of the equilibrium called the *exterior equilibrium* (*cf.* [18] and [20]) with parameters v_i given exogenously. The point (p, q_0, q_1) is called the exterior equilibrium for given influence coefficients (v_0, v_1) if the market is balanced — that is, if equality (1) is valid and the maximum conditions (7) and (9) hold.

In the following, we consider only the case in which the collection of producing participants is fixed (i.e., it does not depend on the values v_i of the influence coefficients). To guarantee this property, we make the following assumption.

A3. For the price $p_0 = b_1$, the following estimate holds:

$$\frac{p_0 - b_0}{a_0} < G(p_0). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

The latter assumption, together with assumptions **A1** and **A2**, ensures that for all nonnegative values of v_i, $i = 0, 1$, there always exists a unique solution of the optimality conditions (7) and (9) satisfying the balance equality (1), that is, the exterior equilibrium. Moreover, conditions (1), (7) and (9) can hold simultaneously if, and only if $p > p_0$, that is, if and only if both outputs q_i, $i = 0, 1$ are strictly positive. The corresponding result was proven in [21].

Lemma 3.1. [21] *Let assumptions* **A1**–**A3** *be valid. If* (p, q_0, q_1) *is an exterior equilibrium state, then* $p > p_0$*, which implies* $q_0 > 0$ *and* $q_1 > 0$*.* ■

We are now able to formulate the main result of this section. We have proven the following theorem in [21]. Its details include a quite long proof, which is available from the authors upon request.

Theorem 3.2. [21] *Under assumptions* **A1–A3***, for any* $D \geq 0$, $v_i \geq 0$, $i = 0,1$, *there uniquely exists an exterior equilibrium state* (p, q_0, q_1) *that depends continuously on the parameters* (D, v_0, v_1). *The equilibrium price* $p = p(D, v_0, v_1)$ *as a function of these parameters is differentiable with respect to both D and* v_0, v_1. *Moreover,* $p(D, v_0, v_1) > p_0$, *and*

$$\frac{\partial p}{\partial D} = \frac{1}{F(\beta, a, v, p)}, \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ (11)$$

where

$$F(\beta, a, v, p) = \frac{1}{a_0 + (1-\beta)v_0} + \frac{v_0 + a_0}{a_0 + (1-\beta)v_0} \cdot \frac{1}{v_1 + a_1} - G'(p). \quad (12)$$

■

4. Interior Equilibrium in Duopoly

Again, following [21], we define the interior equilibrium. We first describe the procedure of verifying the influence coefficients v_i as given in [21]. Assume that we have an exterior equilibrium state (p, q_0, q_1) that has occurred for some v_0, v_1, and D. One of the producers, say, k, temporarily changes its behavior: it abstains from maximizing the conjectured profit (or the convex combination of the domestic social surplus and its net profit, as in case $k = 0$) and makes small fluctuations around its output volume q_k. In mathematical terms, it is tantamount to restricting the model to the monopoly of agent i with $i \neq k$ with the output q_k subtracted from the active demand.

A fluctuation in the supply by agent k is then equivalent to accepting that the active demand varies by $\delta D_k := \delta(D - q_k) \equiv -\delta q_k$. If we consider these variations as infinitesimal, we conclude that by observing the corresponding variations of the equilibrium price, agent k can estimate the derivative of the equilibrium price with respect to the active demand, which *coincides* with his/her own influence coefficient.

Note that in applying Eqs. (11) and (12) from Theorem 3.2 to calculate the derivatives, agent k is (temporarily) absent from the equilibrium model: hence, the terms with $i = k$ must be eliminated from the sum determining $F(\beta, a, v, p)$; moreover, $\beta = 0$ is necessarily when $k = 0$. We thus come to the following criterion.

At the exterior equilibrium (p, q_0, q_1), the influence coefficients v_k, $k = 0,1$, are *consistent* if the following equalities hold:

$$v_0 = \frac{1}{\frac{1}{a_1 + v_1} - G'(p)}, \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ (13)$$

and

$$v_1 = \frac{1}{\frac{1}{a_0 + (1-\beta)v_0} - G'(p)}. \quad . \ . \ . \ . \ . \ . \ . \ . \ (14)$$

We are now ready to define the concept of interior equilibrium.

Definition 4.1. *The collection* (p, q_0, q_1, v_0, v_1), *where* $v_k \geq 0$, $k = 0,1$, *is referred to as an interior equilibrium state if, for the considered influence coefficients, the collection* (p, q_0, q_1) *is the exterior equilibrium and the consistency criterion is satisfied for* v_k, $k = 0,1$.

Remark 4.1. If both agents $i = 0$ and $i = 1$ were net profit-maximizing companies, Eqs. (13) and (14) would be reduced to the uniform equations obtained independently in [19] and [22]:

$$v_i = \frac{1}{\frac{1}{a_j + v_j} - G'(p)}, \quad i, j = 0,1; i \neq j. \quad . \ . \ (15)$$

The following theorem is an extension of Theorem 4.2 from [20] to the case of the above-described mixed duopoly.

Theorem 4.1. [21] *Under assumptions* **A1–A3***, for any* $D \geq 0$, *there exists (at least one) interior equilibrium state.*

Proof. The proof is an obvious extension of that for Theorem 4.2 in [20]. ■

In our further publications, we will extend the obtained results to the case of non-differentiable demand functions. However, some of the necessary techniques must be developed now for the differentiable case. We denote the value of the demand function's derivative by $\tau = G'(p)$ and rewrite the consistency Eqs. (13) and (14) in the following forms:

$$v_0 = \frac{1}{\frac{1}{a_1 + v_1} - \tau}, \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ (16)$$

and

$$v_1 = \frac{1}{\frac{1}{a_0 + (1-\beta)v_0} - \tau}, \quad . \ . \ . \ . \ . \ . \ . \ . \ (17)$$

where $\tau = [-\infty, 0]$. If $\tau = -\infty$, the system (16)–(17) has the unique solution $v_i(\tau) = 0$, $i = 0,1$. The following result was demonstrated in [21].

Theorem 4.2. [21] *Under assumptions* **A1–A3** *and for any* $\tau \in (-\infty, 0]$, *there exists a unique solution of Eqs.* (16) *and* (17), *and this one-to-one correspondence is a continuous function of the variable* τ. *Moreover,* $v_i(\tau) \to 0$ *when* $\tau \to -\infty$, *and* $v_i(\tau)$ *strictly increases up to* $v_i(0)$ *as* τ *grows and tends to zero,* $i = 0,1$. ■

5. Numerical Experiments: Duopoly

To illustrate the difference between the mixed duopoly studied in this paper, the standard mixed and classical

Table 1. Cost function parameters.

Agent i	b_i	a_i
0	2.00	0.02000
1	1.75	0.01750
2	3.25	0.00834

duopoly cases related to the conjectural variations equilibrium with consistent conjectures (influence coefficients) in our previous paper, [21] we applied Eqs. (13) and (14) to a simple example of oligopoly in the electricity market from [20] and [22]. The only difference in our modified example from the instance of [22] was the following: in their case, all six agents (suppliers) were private companies that produced electricity and maximized their net profits. In our example examined in [21], similar to [20], we assumed agent 0 (agent 2 in some instances) to be a public enterprise seeking to maximize the convex combination of the domestic social surplus and its profit, as described in (4), while the other generator was a private firm that maximized its net profit. Similar numerical experiments were conducted and reported in [20], but only for $\beta = 1$. All the other parameters involved in the inverse demand function $p = p(G,D)$ and the producers' cost functions were exactly the same as in [20].

Therefore, following the above-mentioned references, we selected the IEEE two-generator, 30-bus system (*cf.* [22]) to illustrate our analysis. The inverse demand function in the electricity market was accepted in the following form:

$$\begin{aligned} p(G,D) &= 50 - 0.02(G+D) \\ &= 50 - 0.02(q_0 + q_1). \qquad (18) \end{aligned}$$

The parameters of the suppliers' (generators') cost functions are listed in **Table 1**. Here, agents 0, 1, and 2 were combined pairwise in the various examples listed below. In particular, Duopoly 1 included agents 0 (public) and 1 (private), whereas Duopoly 2 included agents 0 (public) and 2 (private).

To find the consistent influence coefficients in the classical duopoly market ($\beta = 0$), [22] used Eq. (15) for both agents, while for the partially mixed duopoly models ($\beta > 0$), we used Eq. (13) for agent 0 and Eq. (14) for agent 1, with $0 < \beta < 1$. Of course, when $\beta = 1$, our model coincided with the mixed duopoly studied in [20]. Having thus obtained the influence coefficients, the (unique) equilibrium was found for Duopolies 1 and 2. The equilibrium results (influence coefficients, production outputs in MWh, equilibrium price, and the objective functions' optimal values in dollars per hour) are presented in **Tables 2–9** from [21]. To make our conjectures v_i comparable to those used in [20], [18], and [22], we divided them by $[-p'(G)] = K^{-1} = 0.02$ and thus obtained $w_i := -v_i/p'(G) = Kv_i = 50v_i$, $i = 0,1$, as shown in **Tables 2** and **6**, where the columns C. and P. contain the influence coefficients for the Cournot and the perfect competition models, respectively.

Table 2. Coefficients of influence w_i for Duopoly 1 (C. = Cournot, P. = Perfect Competition).

i	$\beta = 0$	$\beta = 1/4$	$\beta = 1/2$	$\beta = 3/4$	$\beta = 1$	C.	P.
0	0.5984	0.5945	0.5900	0.5849	0.5789	1.0	0.0
1	0.6152	0.5911	0.5643	0.5341	0.5000	1.0	0.0

Table 3. Consistent equilibrium (production volumes q_i, the total volume G, price p, and the objective functions' values) for Duopoly 1.

	$\beta = 0$	$\beta = 1/4$	$\beta = 1/2$	$\beta = 3/4$	$\beta = 1$
q_0	651.47	746.03	848.83	960.62	1,082.07
q_1	707.20	675.74	641.04	602.67	560.18
G	1,358.67	1,421.77	1,489.87	1,563.29	1,642.25
p	22.83	21.57	20.20	18.73	17.16
S	9,323.80	14,083.76	19,344.35	25,176.38	31,659.88
π_1	10,529.17	9,394.10	8,233.21	7,057.76	5,883.83

Table 4. Cournot equilibrium (production volumes q_i, the total volume G, price p, and the objective functions' values) for Duopoly 1.

	$\beta = 0$	$\beta = 1/4$	$\beta = 1/2$	$\beta = 3/4$	$\beta = 1$
q_0	588.5	711.4	851.4	1,012.6	1,200.00
q_1	634.4	591.7	543.0	486.9	421.7
G	1,222.0	1,303.1	1,394.4	1,499.5	1,621.7
p	25.54	23.94	22.11	20.01	17.57
S	10,390	14,791	19,596	24,847	30,577
π_1	11,571	10,066	8,475	6,817	5,116

Table 5. Perfect competition equilibrium (production volumes q_i, the total volume G, price p, and the objective functions' values) for Duopoly 1.

	$\beta = 0$	$\beta = 1/4$	$\beta = 1/2$	$\beta = 3/4$	$\beta = 1$
q_0	759.10	759.10	759.10	759.10	759.10
q_1	881.80	881.80	881.80	881.80	881.80
G	1,640.90	1,640.90	1,640.90	1,640.90	1,640.90
p	17.18	17.18	17.18	17.18	17.18
S	5,760.81	12,493.01	19,225.21	25,957.42	32,689.62
π_1	6,802.43	6,802.43	6,802.43	6,802.43	6,802.43

Tables 3–5 from [21] demonstrate the numerical results for Duopoly 1.

As **Table 3** shows, the market clearing price (equilibrium price) in the case of the classic duopoly ($\beta = 0$) is quite high, equaling $p_1 = \$22.83$, compared to the mixed duopoly equilibrium price $p_2 = \$17.16$, which is only 75% of the former. The assertions of Theorem 4.2 are also well confirmed: the total production volume grows together with the public firm's output and the domestic social surplus, while the clearing price (as well as the private company's output and net profit) decreases when β increases from 0 to 1. A conclusion is that the higher the proportion of the domestic social surplus in the public firm's objective function, the greater the total production volume, and hence, the lower the clearing price of electricity.

It is also interesting to compare the results in the CVE

Table 6. Coefficients of influence w_i for Duopoly 2 (C. = Cournot, P. = Perfect Competition).

i	$\beta = 0$	1/4	1/2	3/4	$\beta = 1$	C.	P.
0	0.5044	0.4989	0.4929	0.4860	0.4783	1.0	0.0
2	0.6007	0.5778	0.5548	0.5286	0.5000	1.0	0.0

Table 7. Consistent equilibrium (production volumes q_i, the total volume G, price p, and the objective functions' values) for Duopoly 2.

	$\beta = 0$	1/4	1/2	3/4	$\beta = 1$
q_0	618.04	710.29	808.20	911.77	1,020.86
q_1	852.19	815.31	775.57	732.78	686.82
G	1,470.67	1,525.60	1,583.77	1,644.55	1,707.68
p	20.60	19.49	18.33	17.11	15.85
S	7,672.99	13,195.04	19,203.31	25,746.79	32,875.44
π_1	11,753.21	10,467.00	9,183.19	7,916.39	6,684.36

Table 8. Cournot equilibrium (production volumes q_i, the total volume G, price p, and the objective functions' values) for Duopoly 2.

	$\beta = 0$	1/4	1/2	3/4	$\beta = 1$
q_0	554.04	686.42	835.73	1,005.56	1,200
q_1	737.88	683.11	621.34	551.13	471
G	1,291.92	1,369.53	1,457.07	1,556.56	1,671
p	24.16	22.61	20.86	18.87	16.59
S	9,208.79	14,124.02	19,391.53	25,023.08	31,015
π_1	13,159.83	11,278.65	9,331.00	7,341.37	5,353

Table 9. Perfect competition equilibrium (production volumes q_i, the total volume G, price p, and the objective functions' values) for Duopoly 2.

	$\beta = 0$	1/4	1/2	3/4	$\beta = 1$
q_0	579.77	579.77	579.77	579.77	580
q_1	1240.46	1240.46	1240.46	1240.46	1240
G	1,820.23	1,820.23	1,820.23	1,820.23	1,820
p	13.60	13.60	13.60	13.60	13.60
S	3,361.34	11,644.43	19,927.51	28,210.60	36,494
π_1	6,416.53	6,416.53	6,416.53	6,416.53	6,417

with consistent conjectures against the production volumes and profits obtained for the same cases in the classic Cournot-Nash equilibrium (i.e., with $w_i = 1$, $i = 0, 1$). **Table 4** provides the numerical results, with $p_3 = \$25.54$ in the classical duopoly ($\beta = 0$), which is much higher than the market equilibrium price, $p_4 = \$17.57$, in the mixed duopoly ($\beta = 1$).

Again, the total electricity production level is monotone growing as parameter β increases, starting from $G_3 = 1{,}222.0$ MWh when $\beta = 0$ and ending with $G_4 = 1{,}621.7$ MWh for $\beta = 1$. Another interesting observation can be made by comparing **Tables 3** and **4**: when the value of β is small or medium ($\beta \leq 0.75$), both companies have higher objective function values by making use of the Cournot conjectures $w_i = 1$, $i = 0, 1$. However, for β greater than 0.75, the orderings are reversed: by relying on the Bulavsky consistent conjectures calculated by Eqs. (13) and (14) instead of the Cournot–Nash conjectures, both companies improve their results significantly.

We also consider the perfect competition model (see **Table 5**) with $w_i = 0$, $i = 0, 1$, which naturally gives the same results for all values of β and is the best for consumers. However, in our example, this model is a runner-up to the mixed duopoly with consistent conjectures, both in the market price, $p_5 = \$17.18$, and in the total production volume, $G_5 = 1640.90$ MWh. Nevertheless, the domestic social surplus (with $\beta = 1$) is slightly higher in this case (perfect competition), \$32,689.62 per hour, than that in the mixed duopoly with consistent conjectures (also $\beta = 1$), which is \$31,689.62 per hour.

Next, we numerically estimate Duopoly 2, in which the same public company 0 competes with a much stronger private company 2 (see **Table 1** for the parameters). The consistent coefficients of influence computed by (13) and (14) are shown in **Table 6**.

Tables 7 through **9** demonstrate the numerical results for Duopoly 2.

The remarks on the numerical results for Duopoly 2 are similar to those for Duopoly 1. For instance, as shown in **Table 7**, the market clearing price (equilibrium price) in case of the classic duopoly ($\beta = 0$) is extremely high reaching $p_6 = \$20.60$, in comparison with the mixed duopoly equilibrium price, $p_7 = \$15.85$, which is 23% lower. The modes of behavior predicted by Theorem 4.2 and Remark 4.1 are also confirmed: the total production volume grows together with the public firm's output and the domestic social surplus, while the clearing price (as well as the private company's output and net profit) decreases when β grows from 0 to 1. As in Duopoly 1, the higher the proportion of the domestic social surplus in the public firm's objective fiunction, the greater the total production volume, and hence, the lower the clearing price of electricity.

Again, it is worthwhile to compare the results in the CVE with consistent conjectures against the production volumes and profits obtained for the same cases in the classic Cournot equilibrium (i.e., with $w_i = 1$, $i = 0, 1$). **Table 8** presents the numerical results: $p_8 = \$24.16$ in the classical duopoly ($\beta = 0$), which is substantially higher than the market equilibrium price $p_9 = \$16.59$ in the mixed duopoly ($\beta = 1$).

Similar to Duopoly 1, the total electricity production level is monotone growing as parameter β increases, starting from $G_6 = 1{,}470.23$ MWh when $\beta = 0$ and ending with $G_7 = 1{,}707.68$ MWh for $\beta = 1$. A similar feature can found by comparing **Tables 7** and **8**: when the value of β is small or medium ($\beta \leq 0.75$), both companies reach the higher objective function's values by making use of the Cournot conjectures $w_i = 1$, $i = 0, 1$. However, for β greater than 0.75, the orderings are reversed: by relying on the Bulavsky consistent conjectures calculated by Eqs. (13) and (14) instead of the Cournot–Nash conjectures, both companies improve their results significantly.

We also consider the perfect competition model (see

Table 9) with $w_i = 0$, $i = 0,1$, which naturally gives the same results for all values of β and is known to be the best for consumers. Indeed, in contrast to Duopoly 1, in Duopoly 2, the perfect competition results are superior (from consumers' point of view) to those of the mixed duopoly with consistent conjectures, both in the market clearing price, $p_{10} = \$13.60$, and in the total production volume, $G_8 = 1820.23$ MWh. In line with this, the domestic social surplus (with $\beta = 1$) is considerably higher in this case (perfect competition), \$36,493.68 per hour, than that in the mixed duopoly with consistent conjectures (also $\beta = 1$), \$32,875.44 per hour.

Finally, pairwise comparison of **Tables 3** and **7** with **Tables 4** and **8** shows that the latter contain higher total production volumes and lower clearing prices than the former. These results may serve as a good example of how a strong private company may implicitly regulate the market price within a (mixed) duopoly: the stronger the private company, the better the results for consumers.

Remark 5.1. This latter point may appear counterintuitive since in reality, the stronger a private company is, the better its chance for a monopoly leading to negative results for consumers. This may happen in the classical duopoly, but not in the mixed one. Indeed, it is easy to verify that the presence of a public company that strives to maximize the domestic social surplus rather than its net profit completely excludes the possibility of a monopoly by the private company, no matter how strong it is (*cf.* [20]), where assumption **A3** is always valid if $b_1 \le b_0$ — i.e., if the private company is stronger than the public one; this implies that the public company never leaves the market). It is possible because the public company aims to maximize the domestic social surplus rather than its net profit. On the other hand, a strong private company can produce more than a weak one, thus decreasing the market clearing price, which is beneficial for consumers. ■

Remark 5.2. Pairwise comparing **Tables 3** and **4** with **Tables 7** and **8**, an interesting phenomenon is evident. Namely, if both companies are private ($\beta = 0$), the Cournot–Nash equilibrium is more attractive for them than the Bulavsky equilibrium (the consistent conjectural variations equilbrium) since the former provides higher profits to the companies than the latter. However, the relationship is exactly opposite (at least, for the private agent $i = 1$) when company $i = 0$ is public. In this case, the Bulavsky equilibrium is more profitable for the private agent $i = 1$ than the Cournot–Nash equilibrium. At the same time, the energy price is always lower in the Bulavsky equilibrium than in the Cournot–Nash equilibrium. In other words, if the energy market is not (implicitly) regulated by the intervention of a public company that strives to enhance the domestic social surplus, then the Cournot–Nash is better for private suppliers, while the Bulavsky equilibrium is more attractive for consumers because of the lower clearing price. On the other hand, when a public company operates in the market, the Bulavsky equilibrium is better (for both suppliers and consumers) than the Cournot–Nash equilibrium because the profits and supplies are higher in the former than in the latter. ■

Remark 5.3. Reconsider the phenomenon described in Remark 5.2 and suppose that a municipality (government) of locations consuming energy from a market with any value of $\beta \in [0,1]$ is responsible enough to issue subsidies to either (*i*) reduce the Cournot–Nash price for the consumer to the price proposed by the Bulavsky equilibrium or (*ii*) compensate the private suppliers' profit losses whenever they switch to the Bulavsky conjectures instead of using Cournot–Nash ones. Simple calculations demonstrate that option (*ii*) is *always* cheaper for an administration than option (*i*) — that is, if they pay subsidies to the producers they save a great deal compared to the subsidies paid to consumers. Moreover, according to Theorems 4.1 and 4.2, it is possible to show that there exists a (unique) value of the convex combination coefficient $\beta^* \in (0,1)$ such that if the "public" company ($i = 0$) weights the domestic social surplus part of its utility function (4) by the parameter β^*, the private company's profits in the Cournot–Nash equilibrium and in the Bulavsky equilibrium will be equal. The latter implies that when the "nationalization degree" of firm $i = 0$ corresponds to the optimal percentage value $\beta = \beta^*$, there is no need to pay subsidies either to the private supplier (because it is indifferent as to what conjectures to apply, Bulavsky's or the Cournot–Nash) or to consumers (since both suppliers are inclined to produce the Bulavsky equilibrium output, which always generates a lower price for consumers than the Cournot–Nash price). In other words, the stated value of $\beta = \beta^*$ could be interpreted as kind of "optimal" percentage of state-owned shares in the "public" company's assets. ■

6. Conclusion

In this paper, we considered a combined mixed duopoly model with the conjectural variations equilibrium (CVE). The agents' conjectures concerned the price variations that depend on the increase or decrease of their production outputs. We established the existence and uniqueness results for the CVE (called the *exterior equilibrium*) for any set of feasible conjectures. To introduce the notion of an *interior equilibrium state*, we developed a consistency criterion for the conjectures (referred to as influence coefficients) and proved the existence theorem for the interior equilibrium (understood as the CVE with consistent conjectures).

To extend our results to the case of non-differentiable demand functions, we also investigated the behavior of the consistent conjectures dependent upon a parameter representing the demand function's derivative with respect to the market price. Numerical experiments with a small electrical power market were conducted, and some comparisons of the CVE (Bulavsky's conjectures) against the classical Cournot–Nash and perfect competition equilibria were made.

In our future papers, we will examine the qualitative behavior of prices and production outputs when the demand function is not necessarily differentiable, and the cost functions are not quadratic. Moreover, the results outlined in Remarks 5.1, 5.2, and 5.3 will receive mathematically rigorous justification.

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