https://doi.org/10.20965/jaciii.2010.p0256



**Paper:**

# Suppression Effect of $\alpha$-Cut Based Inference on Consequence Deviations

**Kiyohiko Uehara***, **Takumi Koyama***, **and Kaoru Hirota****

*Ibaraki University, 4-12-1 Nakanarusawa-cho, Hitachi 316-8511, Japan
E-mail: k-uehara@mx.ibaraki.ac.jp
**Tokyo Institute of Technology, G3-49 4259 Nagatsuta, Midori-ku, Yokohama 226-8502, Japan
[Received November 16, 2009; accepted January 29, 2010]

**This paper clarifies that inference based on $\alpha$-cut and generalized mean ($\alpha$-GEMII) is effective in suppressing consequence deviations. The suppression effect of $\alpha$-GEMII is numerically evaluated in comparison to conventional inference based on the Compositional Rule of Inference (CRI). CRI-based parallel inference causes discontinuous deviations in the least upper and greatest lower bounds of deduced fuzzy sets even when it models the continuous input-output relation of a system and given facts change continuously. In contrast, $\alpha$-GEMII can suppress the deviations because of its schemes originally developed for constraint propagation control. In simulations, indices are defined for numerically evaluating the degree to which deduced consequences follow the change in fuzzy outputs of given systems. Simulation results show that $\alpha$-GEMII is effective in suppressing the deviations, compared to CRI-based parallel inference. In effective use of the schemes for suppressing the consequence deviations, $\alpha$-GEMII can be applied to nonlinear prediction filters for complex time series, especially with fluctuations that do not always originate from a correlation between time series data.**

**Keywords:** fuzzy inference, convex fuzzy set, $\alpha$-cut, generalized mean, compositional rule of inference

## 1. Introduction

Convex fuzzy sets provide the base to cope with fuzzy numerical information [1] and play a major role in presenting the comprehensibility of fuzzy modeling [2]. Antecedent and consequent parts of fuzzy rules are often defined by convex fuzzy sets [3, 4]. Fuzzy partitioning is also conducted by convex fuzzy sets [5].

The base of fuzzy inference has been presented by the Compositional Rule of Inference (CRI) [6, 7]. The conventional methods of parallel fuzzy inference, based on CRI, may deduce consequences in non-convex forms even if the consequent parts of fuzzy rules are all defined by convex fuzzy sets for representing fuzzy numerical information. Then, it may require to translate the consequences into convex fuzzy sets in order to treat them as fuzzy numbers, especially when the consequent parts of fuzzy rules are sparse [8].

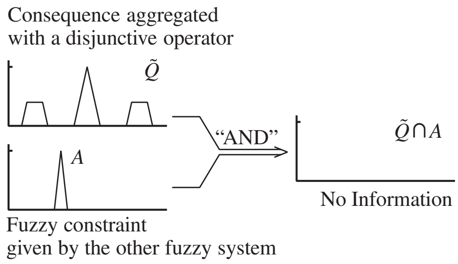



**Fig. 1.** Inference consequence deduced with sparsely placed fuzzy sets and the problem in its use with the other fuzzy system.

**Figure 1** exemplifies the difficulty in using an inference consequence in the non-convex form. In the figure, the conventional fuzzy inference is supposed to be based on the max-min composition and the aggregation with the maximum operation, as an example. The inference consequence $\tilde{Q}$ in the figure consists of sparsely placed fuzzy sets which may stem from sparsely placed consequent parts of the fuzzy rules.

The consequences with sparsely placed fuzzy sets are possibly obtained when functions, to be approximated by fuzzy inference schemes, vary largely at the range where a small number of fuzzy rules are assigned to reduce memory resources or computational cost. They may also be obtained in the course of fuzzy rule adaptation. If a larger number of fuzzy rules can be added among the sparsely placed fuzzy sets of consequent parts, the fuzzy sets may appear among the sparsely placed fuzzy sets in deduced consequences, not outside of them.

In such situation, if it is necessary to satisfy both $\tilde{Q}$ and the fuzzy constraint $A$ given by the other fuzzy system as shown in **Fig. 1**, no information is obtained from the conjunctive operation $\cap$ between $\tilde{Q}$ and $A$. In this case, fuzzy information must implicitly exist between the sparsely placed fuzzy sets in $\tilde{Q}$ and therefore the conjunctive operation $\cap$ between $\tilde{Q}$ and $A$ may provide some information. In order to avoid such problems, inference consequences are required to be convex fuzzy sets so as to interpolate the sparsely placed fuzzy sets in $\tilde{Q}$ while the fuzziness and specificity of the consequences are controlled to overlap

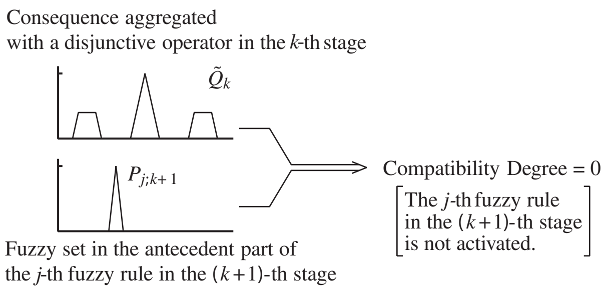



**Fig. 2.** Inference consequence $\tilde{Q}_k$ deduced with sparsely placed fuzzy sets in the $k$-th stage and the problem in its use as a given fact in the $(k+1)$-th stage.

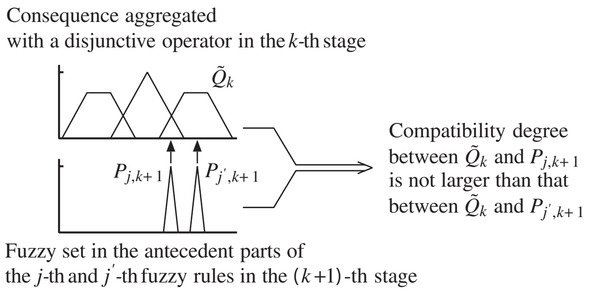



**Fig. 3.** Inference consequence $\tilde{Q}_k$ deduced with overlapped fuzzy sets in the $k$-th stage and the problem in its use as a given fact in the $(k+1)$-th stage.

$\tilde{Q}$ appropriately. The overlapping is to reflect the distribution forms of fuzzy sets in consequent parts of fuzzy rules and compatibility degrees to the forms of deduced consequences.

Moreover, the non-convexity of consequences causes problems in multistage-parallel fuzzy inference based on CRI, especially when fuzzy sets in the consequent parts of fuzzy rules are sparse. **Fig. 2** exemplifies the difficulty in giving a consequence in the non-convex form as a fact to the next stage. In this example, the conventional fuzzy inference is supposed to be based on the max-min composition and the aggregation with the maximum operation. In **Fig. 2**, the inference consequence $\tilde{Q}_k$, deduced in the $k$-th stage, consists of sparsely placed fuzzy sets possibly due to sparsely placed fuzzy sets in the consequent parts of the fuzzy rules in the $k$-th stage.

In such situation as exemplified in **Fig. 2**, the compatibility degree becomes zero between $\tilde{Q}_k$ and the fuzzy set $P_{j,k+1}$ in the antecedent part of the $j$-th fuzzy rule in the $(k+1)$-th stage. As a result, the $j$-th fuzzy rule in the $(k+1)$-th stage is *not* activated. In this case, however, fuzzy information must implicitly exist between the sparsely placed fuzzy sets in $\tilde{Q}_k$ and therefore the conjunctive operation $\cap$ between $\tilde{Q}_k$ and $P_{j,k+1}$ may provide a non-zero compatibility degree which activates the $j$-th fuzzy rule in the $(k+1)$-th stage.

**Figure 3** exemplifies the other viewpoint of the problem in non-convex consequences. In this example, the consequence $\tilde{Q}_k$ consists of fuzzy sets overlapped with each other. As $P_{j,k+1}$ is placed near to the fuzzy set with the largest compatibility degree in $\tilde{Q}_k$, compared with $P_{j',k+1}$, the compatibility degree between $\tilde{Q}_k$ and $P_{j,k+1}$ is expected to be larger than that between $\tilde{Q}_k$ and $P_{j',k+1}$. The non-convex form of consequence $\tilde{Q}_k$, however, causes the compatibility degree between $\tilde{Q}_k$ and $P_{j,k+1}$ *not* to be larger than that between $\tilde{Q}_k$ and $P_{j',k+1}$.

In order to solve the problems of non-convex consequences in multistage-parallel fuzzy inference based on CRI, inference consequences are required to be convex fuzzy sets so as to interpolate the fuzzy sets aggregated in the consequence $\tilde{Q}_k$ while the fuzziness and specificity of the consequences are controlled to overlap $\tilde{Q}_k$ appropriately. The overlapping is to reflect compatibility degrees to the forms of deduced consequences, considering the distribution forms of fuzzy sets in the consequent parts of fuzzy rules.

The considerations on the related works are as follows: V. G. Kaburlasos et al. studied fuzzy interval numbers based on generalized intervals in the lattice theory framework [9]. The generalized intervals can be viewed as an extension of conventional $\alpha$-cuts. They also proposed a metric with tunable nonlinearities [10–12] which provide valuable methods to design fuzzy systems. Then, they proposed novel fuzzy inference methods [10, 11] which may effectively alleviate the curse of dimensionality problems regarding the number of rules in a fuzzy inference system.

S.-Q. Fan et al. proposed the lower and upper approximate fuzzy-inference methods based on the fuzzy concept lattice [13]. They pointed out that the combined use of the two methods will make the fuzzy inference more precise. K. Uehara proposed an inference method on the basis of the weighted average of fuzzy sets. This method provides a learning scheme for fuzzy exemplars while guaranteeing convexity in deduced consequences [14]. Y.-Z. Zhang and H.-X. Li proposed fuzzy inference methods based on variable weighted synthesis [15]. Some models with the methods can be viewed as the extensions of conventional fuzzy inference based on CRI.

The inference methods proposed in the related works mentioned above, however, do not include the control schemes for fuzziness and specificity in deduced consequences while guaranteeing convexity in deduced consequences. Then, the methods cannot reflect the distribution forms of fuzzy sets in the consequent parts of fuzzy rules to the forms of deduced consequences while convexity in deduced consequences is guaranteed. The distribution forms give the weight information of knowledge in the universes of discourse and therefore they are required to be reflected to the forms of deduced consequences.

Z. Huang and Q. Shen proposed fuzzy interpolative reasoning via scale and move transformations [16, 17]. The transformations also contribute to guaranteeing the convexity of deduced consequences in interpolative operations. K. W. Wong et al. proposed fuzzy rule interpolation for multidimensional input spaces [18]. In the fuzzy rule interpolation, deduced consequences can be obtained in convex forms. The interpolative reasoning, however, does not reflect the distribution forms of fuzzy sets in the consequent parts of fuzzy rules to the forms of deduced consequences. This is because its original sake is put on reasoning with sparse fuzzy rules, i.e., the interpolation between the forms of fuzzy sets in the consequent parts of fuzzy rules adjacent to given facts. Therefore, the forms of deduced consequences cannot be reflected by the distribution forms of fuzzy sets in the consequent parts. For example, even if fuzzy sets in consequent parts of activated fuzzy rules are distributed widely in the universes of discourse, fuzziness and specificity of deduced consequences do not become larger and smaller, respectively. Although conventional fuzzy inference, based on CRI, provides such reflection, it cannot guarantee the convexity of deduced consequences as described earlier. In addition, CRI-based parallel inference has the other problems as discussed in the following.

In conventional fuzzy inference based on CRI, fuzzy constraint propagation leads to larger fuzziness and smaller specificity of deduced consequences compared with those of the consequent parts of fuzzy rules. It can be seen as agreeable properties from the viewpoints of human inference. Conventional parallel fuzzy inference, however, tends to cause excessive fuzziness increase and specificity decrease in deduced consequences, which also leads to fuzziness explosion and specificity dispersion in *multistage-parallel* fuzzy inference [19–21]. The problems mentioned above arise from aggregating consequences with disjunctive or conjunctive operators [6, 7, 22]. These operators preserve the positions of fuzzy sets for consequent parts in the process of aggregation.

In order to solve the problems, K. Uehara and K. Hirota have proposed an inference method based on $\alpha$-cuts and generalized mean, named $\alpha$-GEM (an abbreviation for '$\alpha$-level-set and GEneralized-Mean based inference') [23]. This method provides rules of inference for controlling fuzzy constraint propagation. $\alpha$-GEM has the following advantages over conventional inference:

(i) It can control the degree to which the fuzzy constraints of given facts are propagated to those of deduced consequences. It can also provide fuzzy constraint modification in deduced consequences by using the control schemes for the fuzzy constraint propagation. The controllability is effective to deduce consequences, reflecting the distribution forms of fuzzy sets in consequent parts and compatibility degrees.

(ii) It guarantees that consequences are obtained in the forms of normal and convex fuzzy sets when the consequent parts of fuzzy rules are defined by normal and convex fuzzy sets. Thus, the consequences can be treated as fuzzy numbers.

(iii) It conducts inference operations on every $\alpha$-cut independently and then provides efficient inference computations by using hardware constructed in parallel.

Conventional $\alpha$-GEM, however, cannot prove consequences to be deduced in symmetric forms even if fuzzy sets in the consequent parts of fuzzy rules are all symmetric and the required conditions, obtained from axiomatic viewpoints, are satisfied. In order to solve the problem in conventional $\alpha$-GEM, an inference method has been proposed by K. Uehara et al. [8], which can prove to deduce consequences in convex and symmetric forms under the required conditions when fuzzy sets in the consequent parts of fuzzy rules are all convex and symmetric. The inference method is called "$\alpha$-GEMII" in this paper to distinguish it from conventional $\alpha$-GEM.

Moreover, K. Uehara et al. have proposed schemes to automatically determine the degree to which fuzzy constraints of given facts are propagated to those of deduced consequences in $\alpha$-GEMII [24]. Thereby, $\alpha$-GEMII governs the propagation of fuzzy convex constraints from given facts to deduced consequences while it reflects the distribution forms of fuzzy sets in consequent parts to the forms of deduced consequences. Then, K. Uehara et al. have also discussed the theoretical aspects of $\alpha$-GEMII for clarifying its basic properties [25].

This paper points out a further problem in CRI-based inference and clarifies that $\alpha$-GEMII can solve the problem. CRI-based parallel inference causes discontinuous deviations in the least upper and greatest lower bounds of deduced consequences even when it models continuous input-output relation of a system and given facts change continuously. This property of CRI-based parallel inference is an obstacle to modeling in which given systems require continuity in their input-output relations. The schemes in $\alpha$-GEMII can suppress the deviations of the least upper and greatest lower bounds of deduced consequences. They are originally developed for the constraint propagation control. The deviations are referred to as *consequence deviations* in this paper.

The suppression effect of $\alpha$-GEMII on the consequence deviations is numerically evaluated in comparison to CRI-based parallel inference. In order to clarify the background of the evaluations, fuzzy-valued functions, defined by fuzzifying numerical functions, and the mapping of fuzzy sets by the fuzzy-valued functions are described. In relation to the mapping, the deviations of consequences deduced with CRI-based parallel inference are illustrated. Then, the mechanism of suppressing the consequence deviations in $\alpha$-GEMII is explained. It is shown that $\alpha$-GEMII is superior to CRI-based parallel inference in reducing the consequence deviations.

Simulation studies are conducted for evaluating $\alpha$-GEMII and CRI-based parallel inference on the consequence deviations and comparing with each other. Numerical indices are defined for evaluating the degree to which consequences follow the change in fuzzy outputs of given systems. Simulation results show that $\alpha$-GEMII deduces consequences with quite smaller deviations in comparison to CRI-based parallel inference.

The proposed methods are expected to be applied to the modeling of nonlinear systems, especially for estimation of possibility distributions. The authors will apply $\alpha$-

GEMII to prediction filters for complex time series with fluctuations which do not always originate from correlation between the time series data. In the following, a brief discussion is provided on the prediction filters for time series.

K. Uehara and K. Hirota applied the basic scheme of $\alpha$-GEMII, namely with the parameter for the weighted arithmetic mean, to Connection Admission Control (CAC) for Asynchronous Transfer Mode (ATM) networks [26]. If cells, i.e., fixed-length packets, are excessively fed into the networks, cells may be lost in ATM switches. In order to achieve the required Quality-of-Service (QoS), ATM switches judge whether the call from a communication terminal can be accepted or not under terminal-declared conditions of the transmission rates as traffic parameters, e.g., Peak Cell Rate (PCR) and Sustainable Cell Rate (SCR). QoS in ATM networks is often evaluated using the Cell Loss Ratio (CLR).

If CAC is conducted only using the sum of PCR, multiplex efficiency is quite low and then transmission cost becomes very high. Even if the sum of PCR exceeds the bandwidth, some terminals can send cells at the declared PCR, guaranteeing required QoS, when the other terminals happen to send cells at cell rates lower than their declared PCR. Terminals do not always send cells at their declared PCR and the timing of sending cells at the declared PCR depends on the contents of transmitted data. CAC is to achieve high transmission efficiency and guarantee the required QoS at the same time. Namely, CAC has to be performed considering such trade-off between transmission efficiency and the guarantee of the required QoS. Multimedia traffic has various cell-rate fluctuations which are not strictly reflected to traffic parameters. It is quite difficult to strictly represent such various cell-rate fluctuations by numerical traffic-parameters because of their dependence on the contents of transmitted multimedia data, especially motion pictures, music, and voice. Uncertainty thus cannot be avoided in CAC which has to be performed only with declared traffic parameters, e.g., PCR and SCR, which have only a part of information on cell-rate fluctuations. Moreover, the large number of combinations of such various cell-rate fluctuations makes difficult apriori statistical analysis.

In the CAC method, the possibility distributions of CLR are predicted by applying the basic scheme of $\alpha$-GEMII for judging the admission of calls. Fuzzy rules for predicting the possibility distributions of CLR are tuned using observed CLR data. As multimedia traffic with various transmission rates is treated in ATM networks, the observed CLR data disperse even in a class of traffic parameters and thus the observed CLR data can be seen as fluctuated values. In use of the convexity in deduced consequences, the schemes of $\alpha$-GEMII provide an effective way to estimate the possibility distributions of CLR which can lead to the guarantee of the required QoS.

$\alpha$-GEMII can be applied to nonlinear prediction filters for much more complex time series, in effective use of the schemes for estimating possibility distributions in convex forms which can be treated as fuzzy numbers. Namely, it is effective in predicting possibility distributions in complex time series data with fluctuations which do not always originate from the correlation between time series data. Especially, it may be more valuable in case where both input and output of prediction filters are required to be modeled with possibility distributions. The predictions of such possibility distributions may be useful in the planning of electric power supply, climate forecasting, environmental predictions, and so forth.

As CRI-based inference causes deviations in consequences, it deduces excessively large and abrupt changing widths of possibility distributions in use of the prediction and then makes prediction precision low. In contrast, $\alpha$-GEMII has schemes for suppressing such deviations and therefore can more properly deduce possibility distributions without excessively large and abrupt changing widths. In addition, the consequences deduced by $\alpha$-GEMII can be treated as fuzzy numbers whereas those deduced by CRI-based inference are not always convex. More detailed discussions on constructing the prediction filters for the time series with fluctuations will be provided in the future.

This paper is organized as follows: In Section 2, definitions and preliminaries are presented. In Section 3, $\alpha$-GEMII and its control for the fuzzy constraint propagation are introduced. In Section 4, the mapping of fuzzy sets by fuzzy-valued functions are discussed. In relation to the mapping, it is illustrated that CRI-based parallel inference makes consequences deviate even when it models continuous input-output relation of a system and given facts change continuously. Then, it is shown that $\alpha$-GEMII schemes are effective in reducing the deviations of consequences. In Section 5, consequences deduced by $\alpha$-GEMII and CRI-based parallel inference are demonstrated by simulations in order to exemplify the mapping. Then, the consequence deviations are numerically evaluated for both $\alpha$-GEMII and CRI-based parallel inference and compared with each other. Section 6 concludes with the summary.

## 2. Definitions and Preliminaries

The $\alpha$-GEMII method can guarantee the convexity in deduced consequences. The convex fuzzy set is defined as follows:

**Definition 1:** A fuzzy set $A$ in the universe of discourse $X$ is called *convex* if and only if its membership function $\mu_A(x)$ satisfies

$$\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \mu_A(x_1) \wedge \mu_A(x_2), \quad . \quad . \quad (1)$$
$$0 \leq \lambda \leq 1, \quad x_1 \in X, \quad x_2 \in X$$

where $\wedge$ denotes the minimum operation and $X$ is the space of real numbers. ■

The corollary shown below is useful in the following discussions.

**Corollary 1:** If and only if a fuzzy set $A$ is convex, its $\alpha$-cut is convex. ■

In this paper, when membership functions are initially given for defining fuzzy sets, they are represented by continuous functions in the space of real numbers. In this case, when a fuzzy set $A$ is convex in the one-dimensional universe of discourse, its $\alpha$-cut $A_\alpha$ is represented by a closed interval $[x^\ell_\alpha, x^u_\alpha]$ according to Corollary 1. On the other hand, when $\alpha$-cuts are initially given for defining fuzzy sets, they are represented by closed intervals in the space of real numbers.

The rules for the inference process can be translated into the rules governing fuzzy constraint propagation [27, 28]. In the inference process, fuzzy constraints in given facts are propagated to consequences. The fuzzy constraints can be characterized by their fuzziness [29, 30] and specificity [31, 32]. In this paper, fuzziness, specificity, and non-specificity are defined as follows [8, 23, 24, 26, 29, 30, 33]:

**Definition 2:** When a fuzzy set $A$, in the universe of discourse $X$ given by a closed interval $[x^\ell, x^u]$ of real numbers, is normal and convex, and represented by the family of its $\alpha_i$-cuts $A_{\alpha_i}, (i = 1, 2, \ldots, m)$, the fuzziness of $A$ is defined as follows:

$$H_{\mathrm{f}}(A) = \frac{2}{m\mathscr{R}(X)} \sum_{i=1}^{m} |\, w_{\alpha_i} - w^{\mathrm{near}} \,|, \quad \ldots \ldots \quad (2)$$

$$A_{\alpha_i} = [x^\ell_{\alpha_i}, x^u_{\alpha_i}], \quad x^\ell_{\alpha_i} \in X, \quad x^u_{\alpha_i} \in X,$$

$$A^{\mathrm{near}} = [x^\ell_{\alpha^{\mathrm{near}}}, x^u_{\alpha^{\mathrm{near}}}], \quad x^\ell_{\alpha^{\mathrm{near}}} \in X, \quad x^u_{\alpha^{\mathrm{near}}} \in X,$$

$$w_{\alpha_i} = x^u_{\alpha_i} - x^\ell_{\alpha_i}, \quad w^{\mathrm{near}} = x^u_{\alpha^{\mathrm{near}}} - x^\ell_{\alpha^{\mathrm{near}}},$$

$$\mathscr{R}(X) = x^u - x^\ell$$

where if $A_{\alpha_i}$ is null, the value of $w_{\alpha_i}$ is defined as zero for convenience. In the same way, if $A^{\mathrm{near}}$ is null, the value of $w^{\mathrm{near}}$ is defined as zero. The symbol $A^{\mathrm{near}}$ denotes the crisp set nearest to $A$ and is given by the $\alpha$-cut of $A$ at the level $\alpha = \alpha^{\mathrm{near}} = 0.5$. ■

**Definition 3:** When a fuzzy set $A$, in the universe of discourse $X$ given by a closed interval $[x^\ell, x^u]$ of real numbers, is normal and convex, and represented by the family of its $\alpha_i$-cuts $A_{\alpha_i}, (i = 1, 2, \ldots, m)$, the non-specificity of $A$ is defined by

$$H_{\mathrm{ns}}(A) = \frac{1}{m\mathscr{R}(X)} \sum_{i=1}^{m} (x^u_{\alpha_i} - x^\ell_{\alpha_i}) \quad \ldots \ldots \quad (3)$$

where an $\alpha_i$-cut $A_{\alpha_i}$ of $A$ is given by $A_{\alpha_i} = [x^\ell_{\alpha_i}, x^u_{\alpha_i}]$. The specificity of $A$ is defined as follows:

$$H_{\mathrm{s}}(A) = 1 - H_{\mathrm{ns}}(A). \quad \ldots \ldots \quad (4)$$

■

Symmetric and asymmetric fuzzy sets are defined as follows:

**Definition 4:** A fuzzy set $A$ is *symmetric* if and only if the least upper and greatest lower bounds of each of its $\alpha$-cuts are placed symmetrically with respect to the reference point of $A$ for all values of $\alpha$. The reference point $x^\circ$ of $A$ is defined by

$$x^\circ = \frac{x^\ell_{\alpha_{\sup}} + x^u_{\alpha_{\sup}}}{2} \quad \ldots \ldots \quad (5)$$

where $\alpha_{\sup} = \sup_\alpha \{\alpha \mid A_\alpha \neq \phi\}$ and $A_{\alpha_{\sup}} = [x^\ell_{\alpha_{\sup}}, x^u_{\alpha_{\sup}}]$. The symbol $\phi$ denotes the null set. A fuzzy set is called *asymmetric* if and only if it is not symmetric. ■

The generalized mean is defined below, which is used in the operations of $\alpha$-GEMII:

**Definition 5:** Generalized mean $M(\{x_j, p_j\}; \omega)$ is defined by

$$M(\{x_j, p_j\}; \omega) = \left[ \frac{\sum_{j=1}^{n} p_j x_j^\omega}{\sum_{j=1}^{n} p_j} \right]^{\frac{1}{\omega}}, \quad x_j > 0, \quad p_j > 0. \quad (6)$$

■

## 3. $\alpha$-GEMII

In this section, $\alpha$-GEMII and its control for fuzzy constraint propagation are introduced.

### 3.1. Operational Process of $\alpha$-GEMII

The $\alpha$-GEMII method treats the following form of inference:

| | | |
|---|---|---|
| Rule 1 | : | If $x$ is $P_1$ then $y$ is $Q_1$. |
| Rule 2 | : | If $x$ is $P_2$ then $y$ is $Q_2$. |
| ⋮ | | ⋮ |
| Rule $n$ | : | If $x$ is $P_n$ then $y$ is $Q_n$. |
| Given Fact | : | $x$ is $\tilde{P}$. |
| Consequence | : | $y$ is $\tilde{Q}$. |

where $P_j$, $Q_j$, $\tilde{P}$, and $\tilde{Q}$ are fuzzy sets. In this paper, $P_j$, $Q_j$, and $\tilde{P}$ are defined by normal and convex fuzzy sets in $[0, 1]$.

The following shows the operational steps of $\alpha$-GEMII [8]:

Step 1a: Obtain the compatibility degree $\tilde{p}_j$ between $\tilde{P}$ and $P_j$. In particular, the compatibility degree can be calculated by

$$\tilde{p}_j = \sup_x [\mu_{\tilde{P}}(x) \wedge \mu_{P_j}(x)] \quad \ldots \ldots \quad (7)$$

where $\mu_{\tilde{P}}(x)$ and $\mu_{P_j}(x)$ denote the membership functions of $\tilde{P}$ and $P_j$, respectively.

Step 2a: Calculate the weighted arithmetic mean $y^\circ_{\tilde{Q}}$ of the reference points $y^\circ_{Q_j}$ of $Q_j$ by

$$y^\circ_{\tilde{Q}} = M(\{y^\circ_{Q_j}, \tilde{p}_j\}; 1). \quad \ldots \ldots \quad (8)$$

Step 3a: Obtain the closed interval $[y^\ell_{\tilde{Q}\alpha}, y^u_{\tilde{Q}\alpha}]$ as the $\alpha$-cut $\tilde{Q}_\alpha$ of the inference consequence $\tilde{Q}$, by using the following equations:

$$y^u_{\tilde{Q}\alpha} = M(\{y^u_{Q_j\alpha} + (1 - y^\circ_{\tilde{Q}}), \tilde{p}_j\}; \omega(\alpha)) - (1 - y^\circ_{\tilde{Q}}) \quad (9)$$

$$y^\ell_{\tilde{Q}\alpha} = \bar{M}(\{y^\ell_{Q_j\alpha} - y^\circ_{\tilde{Q}}, \tilde{p}_j\}; \omega(\alpha)) + y^\circ_{\tilde{Q}}, \quad \ldots \quad (10)$$

The symbol $\omega(\alpha)$ denotes a non-increasing function of $\alpha$. The function $\bar{M}(\{x_j, p_j\}; \omega)$ is defined by

$$\bar{M}(\{x_j, p_j\};\omega) = 1 - M(\{1-x_j,\ p_j\};\omega). \quad (11)$$

The operation with $\bar{M}$ is called *the dual operation of* $M$ in this paper.

Step 4a: If the need arises, obtain $\tilde{Q}$ from $\tilde{Q}_\alpha$ based on the resolution identity theorem. ■

The non-increasing function $\omega(\alpha)$ can control the propagation in fuzziness and specificity from given facts to deduced consequences. In the next subsection, a scheme is presented to automatically generate $\omega(\alpha)$.

Although $\alpha$ is greater than 0, the value of $\omega(0)$ is interpreted as that of the parameter $\omega$ in the generalized mean, which is used to deduce the least upper and greatest lower bounds of the support set $\tilde{Q}_{\mathrm{supp}}$ of $\tilde{Q}$ in Step 3a [8].

The support set $A_{\mathrm{supp}}$ of a fuzzy set $A$ can be defined by using $\alpha$-cuts as follows [8]:

$$A_{\mathrm{supp}} = \bigcup_{\alpha} A_\alpha. \quad (12)$$

The support sets of fuzzy sets are important for providing the possibility range in the universes of discourse. $\alpha$-GEMII can deduce support sets with the operations exactly same as for deducing $\tilde{Q}_\alpha$ at the level $\alpha > 0$ [8].

### 3.2. Propagation Control for Fuzzy Convex Constraints

The non-increasing function $\omega(\alpha)$ is automatically generated by using the scheme shown below [24]:

Step 1b: Compose the fuzzy rules "If $x$ is $P_j$ then $x$ is $P_j$." by using $P_j, (j = 1,2,\ldots,n)$, in the antecedent parts of initially given fuzzy rules. For convenience, let the composed fuzzy rules be called *Fuzzy Tautological Rules* (FTRs) in this paper.

Step 2b: Perform the inference operations, following Steps 1a–4a with a given fact $\tilde{P}$ and the FTRs. Let the deduced consequence be denoted by $\tilde{P}'$. Tune $\omega(\alpha)$ so as to minimize the difference in fuzziness and specificity between $\tilde{P}$ and $\tilde{P}'$ using the $\alpha$-cuts of $\tilde{P}$ and $\tilde{P}'$, under the conditions shown below:

$$\omega(\alpha') \geq \omega(\alpha''), \quad \alpha' < \alpha'', \quad (13)$$

$$\omega(\alpha) \geq 1. \quad (14)$$

Step 3b: Deduce $\tilde{Q}_\alpha$, following Steps 1a–4a with the tuned $\omega(\alpha)$. ■

Inference properties provided by adopting Eq. (14) are discussed in [24]. Although the other schemes can be conceivable by changing the condition shown in Eq. (14), this paper adopts Eq. (14) for simplicity.

### 3.3. Computational Efficiency

$\alpha$-GEMII is evaluated in terms of computational efficiency in the digital computing environment, compared to CRI-based inference. It should be noted that it is difficult to simply compare with each other in computational efficiency because $\alpha$-GEMII has an advantage over CRI-based inference in deducing convex fuzzy sets. Namely, in order to achieve more proper comparison between them, CRI-based inference is to be considered with some additional computational cost for transforming nonconvex consequences to convex forms. As CRI-based inference does not have such transformation methods, the following evaluations do not include the computational cost of the transformation for CRI-based inference.

First, the order is evaluated for inference with arbitrary forms of membership functions. The computational efficiency of CRI-based inference is evaluated as follows: Let the universes of discourse $X$ and $Y$ be discrete spaces with $n_{\mathrm{s}}$ elements each, considering the digital computing environment, and a membership grade be assigned to each element in $X$ and $Y$. The order for inference operation with each fuzzy rule consists of $O(n_{\mathrm{s}}^2)$ for obtaining the fuzzy relation $R_j$ for the $j$-th fuzzy rule, $O(n_{\mathrm{s}}^2)$ for the minimum operations in composition, and $O(n_{\mathrm{s}}^2)$ for the maximum operations in composition. Then, the inference operation for each fuzzy rule is performed at the order $O(n_{\mathrm{s}}^2)$. Totally, the order of CRI-based inference is $O(n \cdot n_{\mathrm{s}}^2)$ where $n$ is the number of fuzzy rules.

The computational efficiency of $\alpha$-GEMII is evaluated as follows: As $\alpha$-GEMII is $\alpha$-cut based, let the non-zero membership grade be quantized into $m$ levels. The order of $\alpha$-GEMII consists of $O(n \cdot \log_2(m))$ for obtaining compatibility degrees [34] in Step 1a, $O(n)$ for calculation in Step 2a, and $O(n \cdot m)$ for calculation in Step 3a. Then, the order of $\alpha$-GEMII is $O(n \cdot m)$. Therefore, $\alpha$-GEMII is much more efficient than CRI-based inference, considering $n_{\mathrm{s}} \gg n$.

Moreover, $\alpha$-GEMII can be performed on every $\alpha$-cut independently and therefore efficient computation can be achieved by using hardware constructed in parallel. The parallel processing makes the order of $\alpha$-GEMII be $O(n)$. Thus, $\alpha$-GEMII has an advantage over CRI-based inference in computational efficiency nevertheless CRI-based inference cannot always be proved to deduce convex consequences.

The generation of $\omega(\alpha)$ in $\alpha$-GEMII is independent from the order of $\alpha$-GEMII, namely it depends on the adopted optimization algorithm. The optimization is rather easy because the error function is a convex function and therefore no local minimum exists, as can be found in Section 5.1. When the genetic algorithm is used for the optimization, the order in performing $\alpha$-GEMII is $O(n_a \cdot n_g)$ where $n_a$ and $n_g$ are the numbers of chromosomes and generations, respectively. More efficient methods for optimizing $\omega(\alpha)$, including analytical or approximation methods, are for further study.

Next, the order is evaluated when triangular membership functions are adopted for inference. As an example, CRI-based inference is supposed to be based on the max-min composition with the fuzzy relation $R_a$, defined in Subsection 5.2, and the aggregation with the maximum operation. In this case, the order of both $\alpha$-GEMII and CRI-based inference is $O(n)$. CRI-based inference, however, additionally requires the operations for transforming non-convex consequences into convex forms in a certain way for avoiding the problems of non-convex consequences as stated in Section 1.

In the following, some comments are provided from the viewpoint of hardware implementation. The power operation is often adopted for modifying fuzzy sets as the linguistic hedge operation and is also used to generate fuzzy relations for CRI-based inference [35–38]. In the sup-$t$-norm composition for CRI-based inference, $t$-norm operations may require the power operation, e.g., Dombi $t$-norm [39]. Thereby, the power operation may have to be implemented in digital fuzzy processors. Therefore, $\alpha$-GEMII, which uses the power operation, may be performed without any additional hardware in digital fuzzy processors.

In implementation of $\alpha$-GEMII with general-purpose processors, the state-of-the-art technologies of CPUs (Central Processing Units) and GPUs (Graphical Processing Units) may be effectively applied. Even in the home-use computers, multi-core CPUs have been adopted and perform parallel processing. Dual-core and quad-core CPUs are commercially available and 6-core CPUs are coming soon. Moreover, in effective use of fast and parallel operations, GPUs, adopted in home-use computers, can perform general scientific computations, in addition to graphical processing. $\alpha$-GEMII can provide its computing architectures suitable for parallel processing with CPUs and GPUs mentioned above since it performs inference operations on every $\alpha$-cut independently.

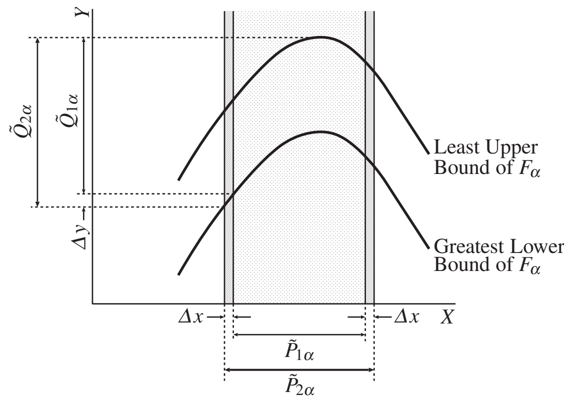



**Fig. 4.** Mapping by a fuzzy-valued function via $\alpha$-cuts.

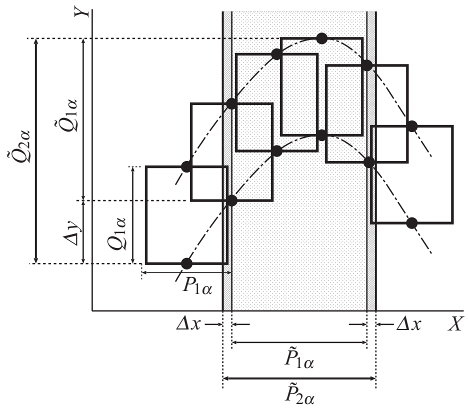



**Fig. 5.** Mapping by CRI-based parallel inference via $\alpha$-cuts.

## 4. Consequence Deviations and Their Suppression

This section discusses the deviations of consequences deduced by CRI-based parallel inference, especially in representing continuous fuzzy-valued functions which are to be defined later. Then, it is clarified that $\alpha$-GEMII schemes have suppression effect on the consequence deviations.

### 4.1. Mapping of Fuzzy Sets by a Fuzzy-Valued Function

When the input-output relation of a given system is fuzzy (rather than numerical), it can be represented by fuzzifying a numerical function. The fuzzified function $F$ is defined by its membership function $\mu_F(x,y)$, $x \in X$, $y \in Y$. Even when the input $x$ of the fuzzified function is a numerical value (a singleton), its output is given by a fuzzy set. In this paper, let the fuzzified function $F$ be called *a fuzzy-valued function*. The mapping of a fuzzy set $\tilde{P}$ to a fuzzy set $\tilde{Q}$ by the fuzzy-valued function $F$ can be performed on the basis of the max-min composition as shown below:

$$\mu_{\tilde{Q}}(y) = \max_x [\mu_{\tilde{P}}(x) \wedge \mu_F(x,y)] \quad . \; . \; . \; . \; . \; . \; . \; (15)$$

where $\mu_{\tilde{P}}(x)$ and $\mu_{\tilde{Q}}(y)$ denote the membership functions of $\tilde{P}$ and $\tilde{Q}$, respectively.

The mapping operation by using $F$ can be conducted at each level of $\alpha$ [34]. Namely, the mapping of the $\alpha$-cut $\tilde{P}_\alpha$ of $\tilde{P}$ to the $\alpha$-cut $\tilde{Q}_\alpha$of $\tilde{Q}$ can be conducted by

$$\mu_{\tilde{Q}_\alpha}(y) = \max_x [\mu_{\tilde{P}_\alpha}(x) \wedge \mu_{F_\alpha}(x,y)] \quad . \; . \; . \; . \; . \; (16)$$

where $\mu_{\tilde{P}_\alpha}(x)$ and $\mu_{\tilde{Q}_\alpha}(y)$ denote the membership functions of $\tilde{P}_\alpha$ and $\tilde{Q}_\alpha$, respectively. The symbol $\mu_{F_\alpha}(x,y)$ means the membership function of the $\alpha$-cut $F_\alpha$ of $F$.

**Figure 4** exemplifies the mapping of given inputs $\tilde{P}_1$ and $\tilde{P}_2$ by $F$ via $\alpha$-cuts. In this example, $F$ is continuous, which means that the least upper and greatest lower bounds of $\mu_{F_\alpha}(x,y)$ with respect to $Y$ are continuous. The symbols $\tilde{P}_{1\alpha}$ and $\tilde{P}_{2\alpha}$ denote the $\alpha$-cuts of $\tilde{P}_1$ and $\tilde{P}_2$, respectively. Moreover, $\tilde{Q}_{1\alpha}$ and $\tilde{Q}_{2\alpha}$ mean the $\alpha$-cuts obtained with the mapping of $\tilde{P}_{1\alpha}$ and $\tilde{P}_{2\alpha}$ by $F_\alpha$, respectively. This figure shows as an example that the small difference $\Delta x$ in the least upper and greatest lower bounds between $\tilde{P}_{1\alpha}$ and $\tilde{P}_{2\alpha}$ provides the small difference $\Delta y$ in the greatest lower bound between $\tilde{Q}_{1\alpha}$ and $\tilde{Q}_{2\alpha}$ because $F$ is continuous.

### 4.2. Mapping of Fuzzy Sets by a Fuzzy-Valued Function Represented with Fuzzy Rules

In system modeling, the capability of representing the input-output relations of given systems is a key to high performance. Parallel fuzzy inference provides a way of representing fuzzy-valued functions with the rule-based scheme, namely by using fuzzy rules. CRI has provided the base of parallel fuzzy inference. CRI-based parallel inference is defined by the following operation in deducing the consequence $\tilde{Q}_j$ with the $j$-th fuzzy rule:

$$\mu_{\tilde{Q}_j}(y) = \max_x[\mu_{\tilde{P}}(x) \wedge \mu_{R_j}(x,y)] \quad \ldots\ldots \quad (17)$$

where $\mu_{R_j}(x,y)$ denotes the membership function of the fuzzy relation $R_j$ which corresponds to the $j$-th fuzzy rule. The symbols $\mu_{\tilde{P}}(x)$ and $\mu_{\tilde{Q}_j}(y)$ mean the membership functions of the given fact $\tilde{P}$ and consequence $\tilde{Q}_j$, respectively. As can be seen in many applications, the membership function $\mu_{R_j}(x,y)$ is often defined by the minimum operation as a $t$-norm. Namely,

$$\mu_{R_j}(x,y) = \mu_{P_j}(x) \wedge \mu_{Q_j}(y), \quad \ldots\ldots\ldots \quad (18)$$

where $\mu_{P_j}(x)$ and $\mu_{Q_j}(y)$ denote the membership functions of $P_j$ and $Q_j$, respectively. When the minimum operation is adopted for generating $\mu_{R_j}(x,y)$, the aggregation of consequences is often conducted by the maximum operation as follows:

$$\mu_{\tilde{Q}}(y) = \max_j \mu_{\tilde{Q}_j}(y) \quad \ldots\ldots\ldots\ldots \quad (19)$$

where $\mu_{\tilde{Q}}(y)$ denotes the membership function of the aggregated consequence $\tilde{Q}$. In modeling, the input-output relation of a given system is represented by a set of $R_j, (j = 1,2,\ldots,n)$. Namely, a fuzzy-valued function discussed in the previous subsection is represented by the set of $R_j$.

Equations (17)–(19) can also be performed via $\alpha$-cuts as follows [34, 40]:

$$\mu_{\tilde{Q}_{j\alpha}}(y) = \max_x[\mu_{\tilde{P}_\alpha}(x) \wedge \mu_{R_{j\alpha}}(x,y)], \quad \ldots \quad (20)$$

$$\mu_{R_{j\alpha}}(x,y) = \mu_{P_{j\alpha}}(x) \wedge \mu_{Q_{j\alpha}}(y), \quad \ldots\ldots \quad (21)$$

$$\mu_{\tilde{Q}_\alpha}(y) = \max_j \mu_{\tilde{Q}_{j\alpha}}(y), \quad \ldots\ldots\ldots \quad (22)$$

where $\mu_{\tilde{P}_\alpha}(x)$, $\mu_{\tilde{Q}_{j\alpha}}(y)$, and $\mu_{R_{j\alpha}}(x,y)$ denote the membership functions of the $\alpha$-cuts of $\tilde{P}$, $\tilde{Q}_j$, and $R_j$, respectively. Moreover, $\mu_{P_{j\alpha}}(x)$ and $\mu_{Q_{j\alpha}}(y)$ show the membership functions of the $\alpha$-cuts of $P_j$ and $Q_j$, respectively. The symbol $\mu_{\tilde{Q}_\alpha}(y)$ means the membership function of the $\alpha$-cut $\tilde{Q}_\alpha$ of $\tilde{Q}$.

When the input-output relation of a given system is to be modeled with a continuous fuzzy-valued function $F$, fuzzy inference is required to be performed with an infinite number of fuzzy rules in the strict sense (without any approximation approaches). As a matter of course, an infinite number of fuzzy rules are not practical and then a finite number of fuzzy rules have to be used in real world applications. In this case, a finite number of fuzzy rules can be seen as representatives of an infinite number of fuzzy rules and are used for approximations in representing continuous fuzzy-valued functions.

Conventional fuzzy inference based on CRI takes into account only possibilities given by fuzzy rules in deducing consequences. Namely, the CRI-based operations have been designed, assuming that no information exists other than given fuzzy rules for deducing possibilities and therefore interpolation is not considered. On this assumption, CRI-based parallel inference can be considered to deduce appropriate consequences. Such inference methods, however, cause problems in deducing consequences as discussed in the next subsection.

### 4.3. Mapping by a Fuzzy-Valued Function with a Finite Number of Fuzzy Rules

This subsection discusses the approximation of a continuous fuzzy-valued function by using a finite number of fuzzy rules. In conventional fuzzy inference based on CRI, a finite number of fuzzy rules cause a problem in the least upper and greatest lower bounds of deduced consequences as described below.

CRI-based parallel inference causes discontinuous deviations in the least upper and greatest lower bounds of deduced fuzzy sets even when it models continuous input-output relation of a system and given facts change continuously. This property of CRI-based parallel inference is an obstacle in modeling given systems whose input-output relations are fuzzy and continuous. In modeling continuous input-output relations with uncertainty, namely representing continuous fuzzy-valued functions, fuzzy inference has to be designed, taking into account interpolative operations with a finite number of fuzzy rules.

In the applications requiring only singleton consequences, the interpolative effect is provided by defuzzification. The operations in defuzzification affect the positions of deduced singletons in the universes of discourse, reflecting the compatibility degrees, since they are performed on numerical values in the universes of discourse. Then, the defuzzification leads to the interpolative effect in aggregating consequences. It can be seen that the interpolative effect of the defuzzification contributes to expanding application fields. CRI-based parallel inference has often been applied in combination with defuzzification techniques. In other words, their combined use conceals the discontinuous deviations of the least upper and greatest lower bounds of consequences aggregated by using disjunctive or conjunctive operators.

The interpolation mentioned above is performed by operations on numerical values in the universes of discourse, whereas CRI is originally conducted by operations on truth values. In the following, CRI-based parallel inference is considered from the viewpoint of the operations on numerical values in the universes of discourse, in which the least upper and greatest lower bounds of deduced consequences are focused on.

When $Q_j, (j = 1,2,\ldots,n)$ are all normal and convex, CRI-based parallel inference deduces the least upper and greatest lower bounds of $\tilde{Q}_\alpha$ which are represented by the following equations, respectively:

$$x^{u}_{\tilde{Q}\alpha} = \max_{j,\,\tilde{p}_j \geq \alpha} [x^{u}_{\tilde{Q}j\alpha}], \qquad (23)$$

$$x^{\ell}_{\tilde{Q}\alpha} = \min_{j,\,\tilde{p}_j \geq \alpha} [x^{\ell}_{\tilde{Q}j\alpha}]. \qquad (24)$$

The symbols $x^{u}_{\tilde{Q}j\alpha}$ and $x^{\ell}_{\tilde{Q}j\alpha}$ denote the least upper and greatest lower bounds of the $\alpha$-cut of $\tilde{Q}_j$, respectively. It should be noted that the discontinuous deviations of $x^{u}_{\tilde{Q}\alpha}$ and $x^{\ell}_{\tilde{Q}\alpha}$ are caused by the two-valued judge weather or not $\tilde{p}_j \geq \alpha$ is satisfied *at each level of* $\alpha$, in selecting operands of the maximum operation in Eq. (23) and the minimum operation in Eq. (24).

**Figure 5** illustrates the inference process based on CRI via $\alpha$-cuts, following Eqs. (20)–(22). In this figure, CRI-based parallel inference at the level $\alpha$ is supposed to represent $F_\alpha$ shown in **Fig. 4**. In **Fig. 5**, the facts $\tilde{P}_1$ and $\tilde{P}_2$ are given via their $\alpha$-cuts $\tilde{P}_{1\alpha}$ and $\tilde{P}_{2\alpha}$, respectively. Six squares in the figure show the $\alpha$-cuts $R_{j\alpha}$ of the fuzzy relations $R_j$ constructed from $P_j$ and $Q_j$, $(j = 1, 2, \ldots, 6)$. For example, the square at the far left in the figure depicts $R_{1\alpha}$ which is constructed with $P_{1\alpha}$ and $Q_{1\alpha}$. The symbols $P_{1\alpha}$ and $Q_{1\alpha}$ denote the $\alpha$-cut of $P_1$ in the antecedent part and the $\alpha$-cut of $Q_1$ in the consequent part of the first fuzzy rule $(j = 1)$, respectively. The symbols $\tilde{Q}_{1\alpha}$ and $\tilde{Q}_{2\alpha}$ mean the $\alpha$-cuts of the consequences deduced with $\tilde{P}_{1\alpha}$ and $\tilde{P}_{2\alpha}$, respectively.

Although the value of $\Delta x$ in **Fig. 5** is the same as that in **Fig. 4**, the value of $\Delta y$ in **Fig. 5** is much larger than that in **Fig. 4**. It stems from the maximum operation for aggregating consequences deduced with a finite number of fuzzy rules. Namely, **Fig. 5** illustrates abrupt change, characterized by Eq. (24), in the greatest lower bound of the $\alpha$-cut of the consequence.

When CRI-based parallel inference *without* defuzzification is applied to control systems or adaptive systems, especially including feedback loops, the consequence deviations may cause the other deviations in the systems. Thus, the consequence deviations may make the systems unstable.

## 4.4. Suppression Effect of $\alpha$-GEMII on Consequence Deviations

In contrast to CRI-based parallel inference, $\alpha$-GEMII has interpolative schemes for fuzzy sets in the consequent parts of fuzzy rules and, at the same time, includes the control schemes for fuzzy constraint propagation. In the operational process of $\alpha$-GEMII, deduced consequences are not directly affected by the two-valued judge characterized by Eqs. (23) and (24).

The control schemes for fuzzy constraint propagation in $\alpha$-GEMII, which follow Steps 1b–3b, contribute to suppressing the consequence deviations. **Fig. 6** shows the mechanism of the suppression. In this figure, M. G. stands for membership grade. This abbreviation is also used in the other figures. All fuzzy sets are defined by triangular membership functions for simplicity. In this case, the values of $\omega(\alpha)$ can be defined by the equations $\omega(1) = 1$

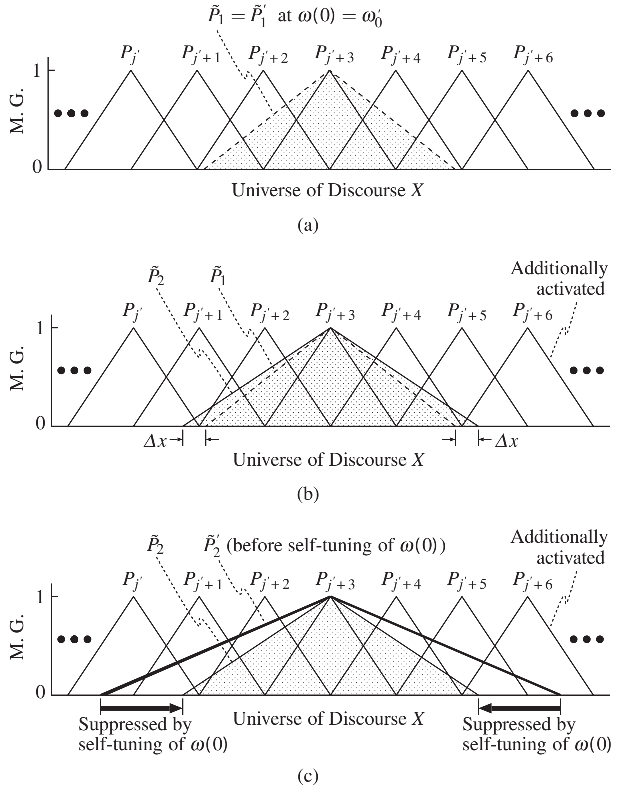



**Fig. 6.** Suppression of consequence deviations by adjusting $\omega(\alpha)$ : An example with $\omega(0)$.

and $\omega(0) = \omega_0$. The value of $\omega_0$ is to be tuned by following Steps 1b–3b. In deducing the support set of $\tilde{Q}$ with the tuned value of $\omega_0$, $\alpha$-GEMII performs the operations exactly same as for the $\alpha$-cuts of the consequences at the level $\alpha > 0$. Thus, the discussions here can be conducted without loosing generality. See Subsection 5.1 for more detail.

In **Fig. 6(a)**, a given fact $\tilde{P}_1$ activates the fuzzy rules with $P_j, (j = j' + 1, j' + 2, \ldots, j' + 5)$ in the antecedent parts. In this figure, the consequence $\tilde{P}_1'$ deduced with $\tilde{P}_1$ and FTRs is supposed to be made equal to $\tilde{P}_1$ by the self-tuning of $\omega(0)$. The tuned value of $\omega(0)$ is denoted by $\omega_0'$ in the figure.

Let the other fact $\tilde{P}_2$ be given as shown in **Fig. 6(b)**. The support set of $\tilde{P}_2$ is wider in its range than that of $\tilde{P}_1$. The small difference $\Delta x$ in the least upper and greatest lower bounds of the support sets is given between $\tilde{P}_1$ and $\tilde{P}_2$. Thereby, $\tilde{P}_2$ activates the fuzzy rules with $P_{j'}$ and $P_{j'+6}$ in addition to the fuzzy rules with $P_j, (j = j' + 1, j' + 2, \ldots, j' + 5)$. In this figure, $\tilde{P}_2'$ deduced with $\tilde{P}_2$ and FTRs is not depicted for simplicity.

For the following explanations, $\tilde{P}_2'$ in **Fig. 6(c)** shows the consequence deduced with $\tilde{P}_2$ and FTRs by using $\omega_0'$ as the value of $\omega(0)$. Here, it should be noted that $\omega_0'$ gives the value originally tuned for $\tilde{P}_1$. Thus, $\tilde{P}_2'$ in this figure is to be adjusted by the self-tuning of $\omega(0)$ so as to minimize the difference between $\tilde{P}_2$ and $\tilde{P}_2'$. Since the fuzzy rules with $P_{j'}$ and $P_{j'+6}$ are additionally activated, the value $\omega_0'$ tuned for $\tilde{P}_1$ makes wider the range of the support set of $\tilde{P}_2'$. The value $\omega_0'$ may abruptly widen the range of $\tilde{Q}_{\mathrm{supp}}$ if $\omega(0)$ is not tuned before deducing consequences for $\tilde{P}_2$. The control schemes for fuzzy constraint propagation in $\alpha$-GEMII, however, make the value of $\omega(0)$ smaller than $\omega_0'$ before deducing consequences for $\tilde{P}_2$, so as to minimize the difference between $\tilde{P}_2$ and $\tilde{P}_2'$ following Steps 1b–3b. As a result, the tuned value of $\omega(0)$ for $\tilde{P}_2$ alleviates the abrupt widening of the range of $\tilde{Q}_{\mathrm{supp}}$ and therefore contributes to suppressing the consequence deviations. It effectively works for the suppression, together with the interpolative effect in $\alpha$-GEMII.

More precise analysis, including mathematical analysis, on the consequence deviations, is for further study. This paper focuses on the numerical evaluations of the suppression effect on consequence deviations as shown in the following.

## 5. Simulation Studies

Simulations are performed to exemplify the mapping described in Section 4, and numerically evaluate the consequence deviations in $\alpha$-GEMII and CRI-based parallel inference.

### 5.1. Self-Tuning for Propagation Control of Fuzzy Convex Constraints

In the simulations, fuzzy sets are all defined by the triangular forms of membership functions. The triangular membership functions are also practically important since they contribute to high computational efficiency in inference operations.

Each triangular membership function is parameterized by the $\alpha$-cut at $\alpha = 1$ and the least upper and greatest lower bounds of the support set [8]. Since the fuzzy sets are all defined by triangular membership functions, the least upper bound of the $\alpha$-cut of each fuzzy set is equal to its greatest lower bound at $\alpha = 1$. In this case, the value of $\omega(1)$ must be 1, which also means $\omega(1)$ is not needed to be tuned, because the least upper bound of the $\alpha$-cut of a deduced fuzzy set has to be equal to its greatest lower bound at $\alpha = 1$. Then, the values of $\omega(\alpha)$ can be defined by the equations $\omega(1) = 1$ and $\omega(0) = \omega_0$. The value of $\omega_0$ is to be tuned for controlling the propagation of the fuzzy convex constraints by following Steps 1b–3b, which determines the fuzziness and specificity of deduced consequences. Namely, the minimization of the difference in fuzzy constraints between $\tilde{P}$ and $\tilde{P}'$ is translated into the tuning of $\omega_0$ so as to minimize the difference in shape of membership functions between them.

For tuning $\omega_0$ so as to minimize the difference in fuzzy constraints between $\tilde{P}$ and $\tilde{P}'$, the objective function shown below is minimized.

$$E = \frac{1}{2}\left\{\left[s\left(x^{\ell}_{\tilde{P}'_{\underline{\alpha}}}\right) - x^{\ell}_{\tilde{P}_{\underline{\alpha}}}\right]^2 + \left[s\left(x^{u}_{\tilde{P}'_{\underline{\alpha}}}\right) - x^{u}_{\tilde{P}_{\underline{\alpha}}}\right]^2\right\}. \quad (25)$$

Here, $x^{u}_{\tilde{P}_{\underline{\alpha}}}$ and $x^{\ell}_{\tilde{P}_{\underline{\alpha}}}$ denote the least upper and greatest lower bounds of the support set of $\tilde{P}$, respectively. The symbols $x^{u}_{\tilde{P}'_{\underline{\alpha}}}$ and $x^{\ell}_{\tilde{P}'_{\underline{\alpha}}}$ mean the least upper and greatest lower bounds of the support set of $\tilde{P}'$, respectively. Moreover,

$$s(x) = x - (x^{\circ}_{\tilde{P}'} - x^{\circ}_{\tilde{P}}). \quad (26)$$

where $x^{\circ}_{\tilde{P}}$ and $x^{\circ}_{\tilde{P}'}$ mean the reference points of $\tilde{P}$ and $\tilde{P}'$, respectively. The function $s(x)$ in Eq. (26) adjusts the position of $\tilde{P}'$ so as to make $x^{\circ}_{\tilde{P}'}$ equal to $x^{\circ}_{\tilde{P}}$ for simplifying the evaluation of the difference in fuzzy constraints between $\tilde{P}$ and $\tilde{P}'$ as shown in Eq. (25).

In order to optimize the value of $\omega_0$, the Genetic Algorithm (GA) was applied. The chromosomes correspond to alternatives for the optimized value of $\omega_0$. The objective function defined by Eq. (25) was used for the evaluations of the chromosomes. The elite preservation strategy is conducted on the basis of the evaluation results obtained by Eq. (25).

### 5.2. Simulation Conditions

The simulations were performed under the following conditions:

(i) Fuzzy sets were all defined by the triangular forms of membership functions as described in Subsection 5.1. In this case, it is to be noted that their reference points are equal to their $\alpha$-cuts at $\alpha = 1$.

(ii) As all fuzzy sets were defined by the triangular membership functions, the support set $\tilde{Q}_{\mathrm{supp}}$ of $\tilde{Q}$ was used for evaluating the consequence deviations. In deducing the support sets of consequences, $\alpha$-GEMII performs the operations exactly same as for the $\alpha$-cuts of the consequences at the level $\alpha > 0$. Therefore, the evaluations can be conducted without losing generality.

(iii) The number $n$ of fuzzy rules was set to 21. The antecedent parts of the fuzzy rules and given facts were defined by symmetric fuzzy sets in $[0,1]$. Each reference point $x^{\circ}_j$ of $P_j$ was set by $x^{\circ}_j = [1/(n-1)]\cdot(j-1) = 0.05\cdot(j-1)$ for the $j$-th fuzzy rule. Namely, the reference points were equally placed in $X$. The least upper bound $x^{u}_{P_{j\underline{\alpha}}}$ and greatest lower bound $x^{\ell}_{P_{j\underline{\alpha}}}$ of the support set of $P_j$ were given so as to satisfy $x^{u}_{P_{j\underline{\alpha}}} - x^{\ell}_{P_{j\underline{\alpha}}} = 2/(n-1) = 0.1$. **Fig. 7** shows the membership functions $\mu_{P_j}(x)$ of $P_j$, $(j = 1, 2, \ldots, 21)$.

(iv) The consequent parts of the fuzzy rules were represented by symmetric fuzzy sets in $[0,1]$. The value of the reference point $y^{\circ}_j$ of $Q_j$ was assigned by $q(x^{\circ}_j)$. The function $q(x)$ is provided by the following equations:

$$q_{\mathrm{lnr}}(x) = 0.8x + 0.1, \quad (27)$$

$$q_{\mathrm{sgm}}(x) = \frac{0.5}{1 + \exp(-20x + 14)} + 0.45, \quad (28)$$

$$q_{\cos}(x) = -0.25\cos(2\pi x) + 0.3. \quad (29)$$

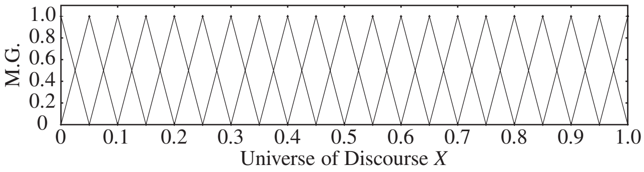
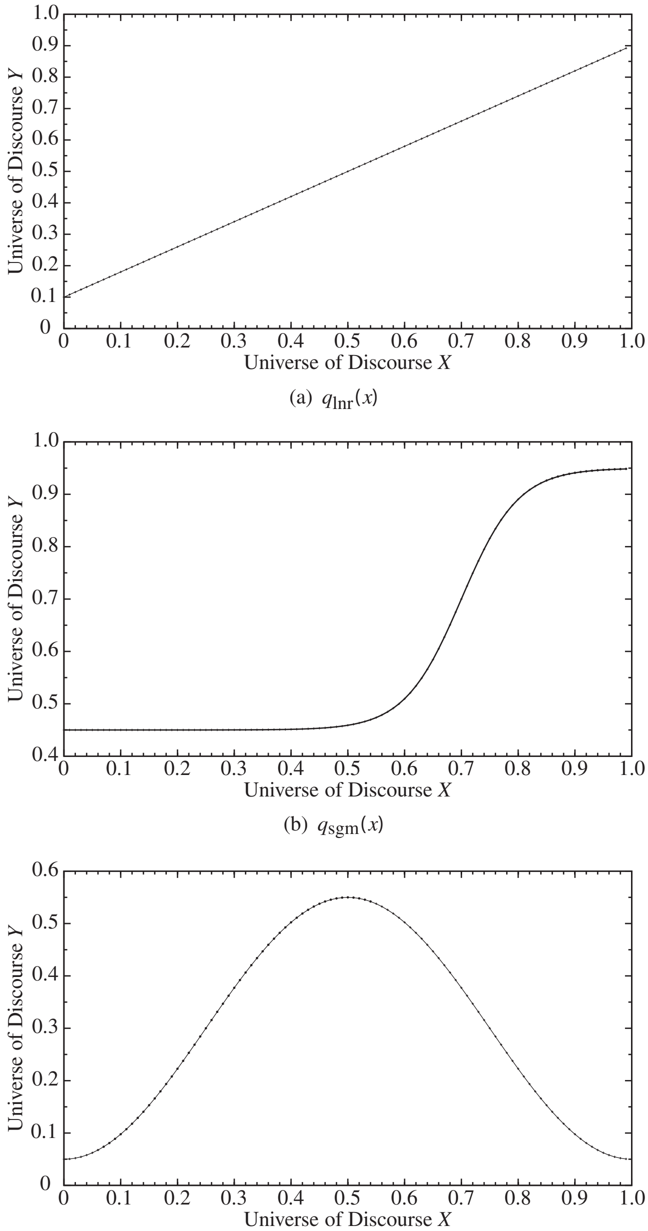



**Fig. 7.** Membership functions for antecedent parts used in simulations.




(a) $q_{\text{lnr}}(x)$

(b) $q_{\text{sgm}}(x)$

(c) $q_{\cos}(x)$

**Fig. 8.** Functions used for placing consequent parts of fuzzy rules.

**Fig. 8** depicts the functions $q_{\text{lnr}}(x)$, $q_{\text{sgm}}(x)$, and $q_{\cos}(x)$. Eqs. (27)–(29) are determined so as to contribute to evaluating the deviations of the least upper and greatest lower bounds of $\tilde{Q}_{\text{supp}}$ in the mapping characterized by the fuzzy rules. The least upper bound $y^{u}_{Q_{j\underline{\alpha}}}$ and greatest lower bound $y^{\ell}_{Q_{j\underline{\alpha}}}$ of the support set of $Q_j$ were given so as to satisfy $y^{u}_{Q_{j\underline{\alpha}}} - y^{\ell}_{Q_{j\underline{\alpha}}} = 0.1$.

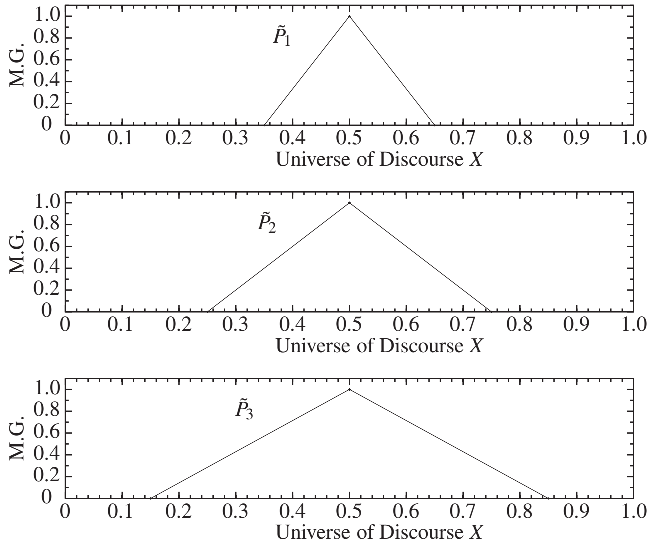



**Fig. 9.** Given facts used in simulations.

(v) Given facts $\tilde{P}$ were all symmetric fuzzy sets and defined by triangular membership functions. Their reference points were all placed at 0.5. In order to evaluate the deviations of the least upper and greatest lower bounds of $\tilde{Q}_{\text{supp}}$, the support-set width $\tilde{w}_{\text{supp}}$ of $\tilde{P}$ were increased by $0.2 \times 10^{-2}$ from 0.1 to 1.0. The support-set width $\tilde{w}_{\text{supp}}$ is defined by $x^{u}_{\tilde{P}_{\underline{\alpha}}} - x^{\ell}_{\tilde{P}_{\underline{\alpha}}}$ where $x^{u}_{\tilde{P}_{\underline{\alpha}}}$ and $x^{\ell}_{\tilde{P}_{\underline{\alpha}}}$ denote the least upper and greatest lower bounds of the support set $\tilde{P}_{\text{supp}}$ of $\tilde{P}$, respectively. Totally, 451 facts were given.

(vi) In tuning $\omega_0$ with GA, the chromosome with 16 bits was used for representing an alternative of the value for $\omega_0$. The number of the chromosomes was fifty. Ten of them were selected as parent chromosomes for next generation by using the elite preservation strategy. Two of the parent chromosomes were randomly selected and they created two offspring chromosomes by the one-point crossover. The crossover was performed at the midpoint of chromosomes, namely between the eighth and ninth bits of each chromosome. Two of the 16 bits in each offspring chromosome were randomly selected and mutated at the probability of 0.3.

(vii) In evaluating the deviations of $\tilde{Q}_{\text{supp}}$, $\alpha$-GEMII was compared to CRI-based parallel inference. In performing CRI, the fuzzy relation $R_a$, the max-min composition, and the aggregation with the maximum operation were adopted, which have been applied in a wide variety of fields. The fuzzy relation $R_a$ is defined by its membership function $\mu_{R_a}(x,y) = \mu_P(x) \wedge \mu_Q(y)$ where $\mu_P(x)$ and $\mu_Q(y)$ denote membership functions for the antecedent and consequent parts of each fuzzy rule, respectively.

## 5.3. Simulation Results

### 5.3.1. Consequences as Mapped Fuzzy Sets

**Figures 10**, **11**, and **12** illustrate consequences deduced in the simulations. The consequences are considered to approximate to the fuzzy sets mapped by fuzzy-valued functions generated by using $q(x)$. These consequences were deduced with the given facts $\tilde{P}_k, (k=1,2,3)$ shown in **Fig. 9**. The reference point of $\tilde{P}_k$ was set to 0.5 whereas the support-set widths of $\tilde{P}_k, (k=1,2,3)$ were set to 0.3, 0.5, and 0.7, respectively. In **Figs. 10**, **11**, and **12**, $\tilde{Q}_{k(\text{new})}$ and $\tilde{Q}_{k(\text{cnv})}$ denote the consequences deduced by $\alpha$-GEMII and CRI-based parallel inference, respectively. The consequences $\tilde{Q}_{k(\text{new})}, (k=1,2,3)$ were deduced with $\tilde{P}_k, (k=1,2,3)$, respectively, and $\tilde{Q}_{k(\text{cnv})}, (k=1,2,3)$ were deduced with $\tilde{P}_k, (k=1,2,3)$, respectively.

**Figure 10** shows the consequences deduced with $Q_j$ assigned by using $q_{\text{lnr}}(x)$. Since $q_{\text{lnr}}(x)$ is a linear function, $\alpha$-GEMII deduced symmetric fuzzy sets. The precise discussions on symmetricity in consequences are provided in [8]. Although CRI-based parallel inference also deduced symmetric fuzzy sets, the deduced fuzzy sets were not convex.

**Figures 11** and **12** depict the consequences deduced with $Q_j$ assigned by using $q_{\text{sgm}}(x)$ and $q_{\cos}(x)$, respectively. In contrast to the consequences shown in **Fig 10**, asymmetric fuzzy sets were deduced, reflecting the nonlinearity of $q_{\text{sgm}}(x)$ and $q_{\cos}(x)$. As can be seen in **Fig. 11**, the possibility in the range upper than the reference point $\tilde{Q}^{\circ}_{k(\text{new})}$ of $\tilde{Q}_{k(\text{new})}$ is larger than that in the range lower than $\tilde{Q}^{\circ}_{k(\text{new})}$. **Fig. 12** shows that the possibility in the range lower than $\tilde{Q}^{\circ}_{k(\text{new})}$ is larger than that in the range upper than $\tilde{Q}^{\circ}_{k(\text{new})}$. Although CRI-based parallel inference also deduced asymmetric fuzzy sets, the deduced fuzzy sets were not convex.

### 5.3.2. Evaluations of Consequence Deviations

The deviations of the least upper and greatest lower bounds of $\tilde{Q}_{\text{supp}}$ were evaluated in changing the support-set width of $\tilde{P}$. For numerical evaluations, the following indices are defined.

$$D_{\text{avr}} = \sum_i [D^u_g(i) + D^\ell_g(i)]/2m_d, \quad (30)$$

$$D_{\max} = \max_i [D^u_g(i), D^\ell_g(i)]. \quad (31)$$

Here, each symbol is defined as follows for the $i$-th data ($i = 1,2,\ldots,m_d$) obtained in the simulations:

$$D^u_g(i) = |\, g^u_{\tilde{Q}}(i) - g^u_q(i)\,|, \quad (32)$$

$$D^\ell_g(i) = |\, g^\ell_{\tilde{Q}}(i) - g^\ell_q(i)\,|, \quad (33)$$

$$g^u_{\tilde{Q}}(i) = \Delta y^u_{\tilde{Q}_{\text{supp}}}(i)/\Delta\tilde{w}_{\text{supp}}(i), \quad (34)$$

$$g^\ell_{\tilde{Q}}(i) = \Delta y^\ell_{\tilde{Q}_{\text{supp}}}(i)/\Delta\tilde{w}_{\text{supp}}(i), \quad (35)$$

$$g^u_q(i) = \Delta y^u_q(i)/\Delta\tilde{w}_{\text{supp}}(i), \quad (36)$$

$$g^\ell_q(i) = \Delta y^\ell_q(i)/\Delta\tilde{w}_{\text{supp}}(i). \quad (37)$$

In Eqs. (34)–(37), $\Delta\tilde{w}_{\text{supp}}(i)$ means the increase of $\tilde{w}_{\text{supp}}(i)$ which is defined by

$$\tilde{w}_{\text{supp}}(i) = x^u_{\tilde{P}_{\text{supp}}}(i) - x^\ell_{\tilde{P}_{\text{supp}}}(i). \quad (38)$$

In Eq. (38), $x^u_{\tilde{P}_{\text{supp}}}(i)$ and $x^\ell_{\tilde{P}_{\text{supp}}}(i)$ denote the least upper and greatest lower bounds of $\tilde{P}_{\text{supp}}$, respectively. In the simulations, the value of $\Delta\tilde{w}_{\text{supp}}(i)$ was set to $0.2 \times 10^{-2}$. The symbols $\Delta y^u_{\tilde{Q}_{\text{supp}}}(i)$ and $\Delta y^\ell_{\tilde{Q}_{\text{supp}}}(i)$ mean the increase of the least upper and greatest lower bounds of $\tilde{Q}_{\text{supp}}$, respectively, when $\tilde{w}_{\text{supp}}(i)$ increases by $\Delta\tilde{w}_{\text{supp}}(i)$. Namely, $g^u_{\tilde{Q}}(i)$ and $g^\ell_{\tilde{Q}}(i)$ show the gradient of the least upper and greatest lower bounds of $\tilde{Q}_{\text{supp}}$, respectively. Moreover, $\Delta y^u_q(i)$ and $\Delta y^\ell_q(i)$ denote the increase of the least upper and greatest lower bounds of $\tilde{Q}_q$, respectively, in increasing $\tilde{w}_{\text{supp}}(i)$ by $\Delta\tilde{w}_{\text{supp}}(i)$, where $\tilde{Q}_q$ is the closed interval obtained by the mapping of $\tilde{P}_{\text{supp}}$ by $q(x)$. Then, $g^u_{\tilde{Q}}(i)$ and $g^\ell_{\tilde{Q}}(i)$ denote the gradient of the least upper and greatest lower bounds of $\tilde{Q}_q$, respectively. Therefore, $D^u_g(i)$ and $D^\ell_g(i)$ mean the difference in the gradient of the least upper and greatest lower bounds, respectively, between $\tilde{Q}_{\text{supp}}$ and $\tilde{Q}_q$. Thereby, they indicate the measures to which the rule-based inference represents the fuzzy-valued functions generated with $q(x)$. The lower values of $D_{\text{avr}}$ and $D_{\max}$ indicate better performance for representing the fuzzy-valued functions.

**Figures 13**, **14**, and **15** show the least upper and greatest lower bounds of $\tilde{Q}_{\text{supp}}$ when $Q_j$ are assigned by using $q_{\text{lnr}}(x)$, $q_{\text{sgm}}(x)$, and $q_{\cos}(x)$, respectively. In these figures, L.U.B. and G.L.B. stand for least upper bound and greatest lower bound, respectively. The solid lines imply the least upper and greatest lower bounds of $\tilde{Q}_{\text{supp}}$ whereas the dotted lines the least upper and greatest lower bounds of $\tilde{Q}_q$. As shown in these figures, the values of both $D_{\text{avr}}$ and $D_{\max}$ indicate that $\alpha$-GEMII provides better performance than CRI-based parallel inference. Especially, $D_{\max}$ proves that $\alpha$-GEMII follows the change of the bounds of $\tilde{Q}_q(x)$ much more properly than CRI-based parallel inference.

As can be found in **Fig. 14**, the least upper bounds of $\tilde{Q}_{\text{supp}}$, deduced by $\alpha$-GEMII, deviate in quite a narrow range in comparison to CRI-based parallel inference and then $\alpha$-GEMII gives much smaller value of $D_{\max}$ than CRI-based parallel inference. In the range upper around 0.5 of $\tilde{w}_{\text{supp}}$ shown in **Fig. 14**, $\alpha$-GEMII properly follows

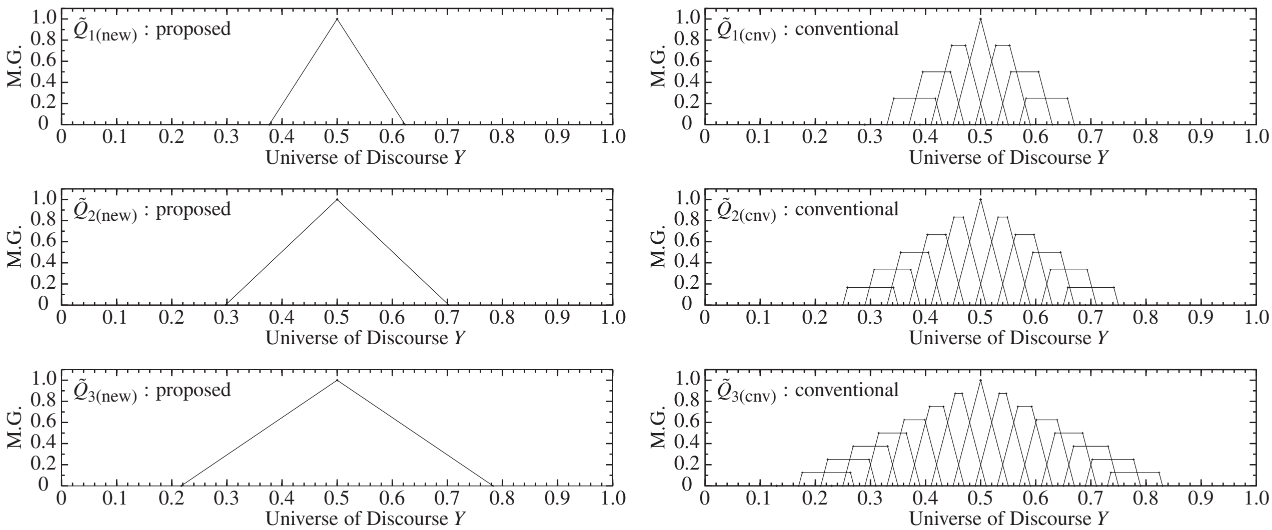



**Fig. 10.** Inference consequences deduced by using fuzzy rules with consequent parts placed by $q_{\mathrm{lnr}}(x)$.

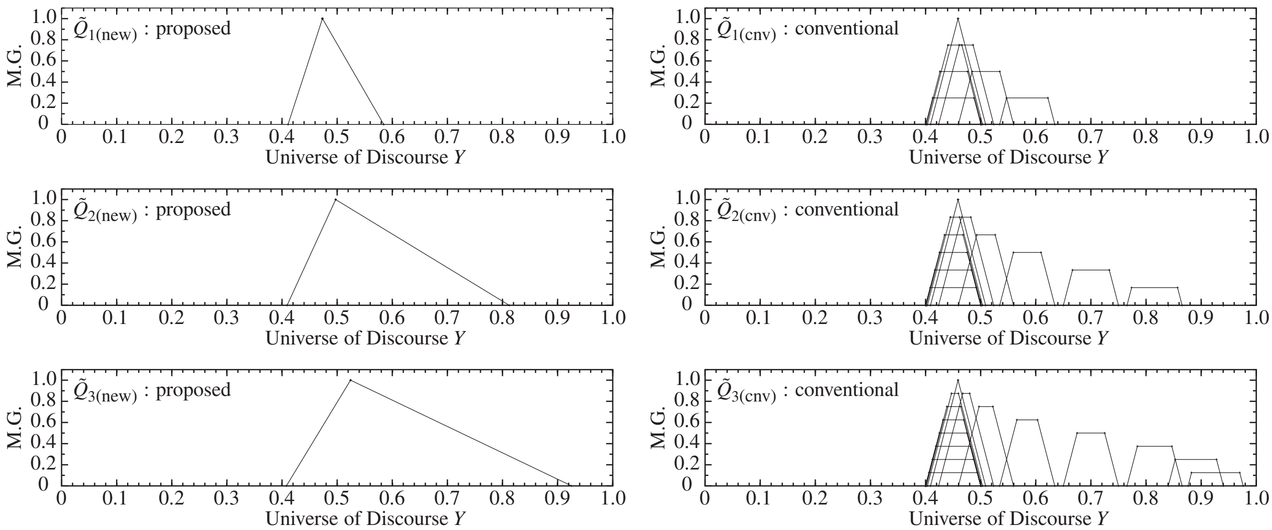



**Fig. 11.** Inference consequences deduced by using fuzzy rules with consequent parts placed by $q_{\mathrm{sgm}}(x)$.

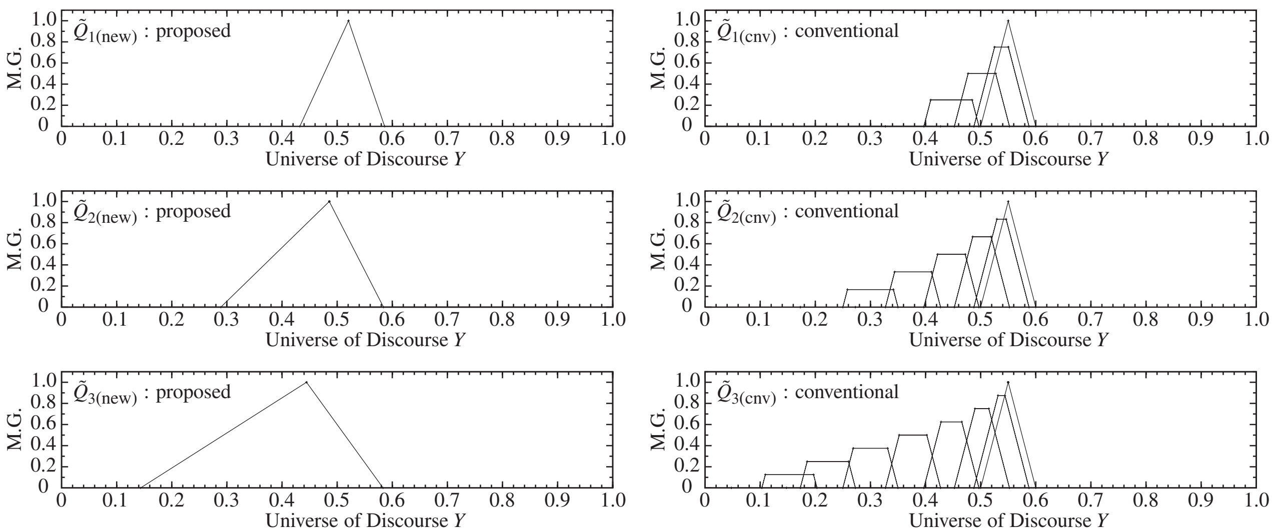
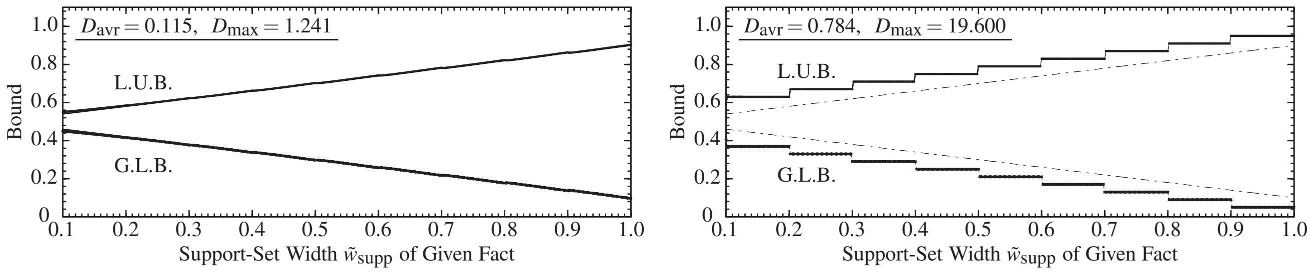
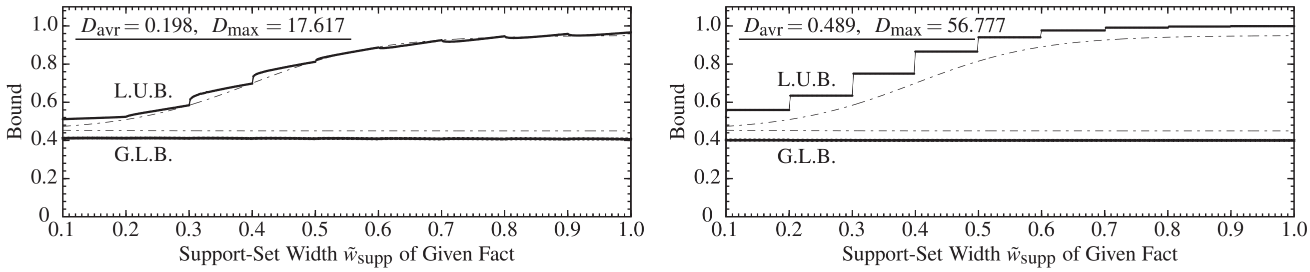



**Fig. 12.** Inference consequences deduced by using fuzzy rules with consequent parts placed by $q_{\cos}(x)$.



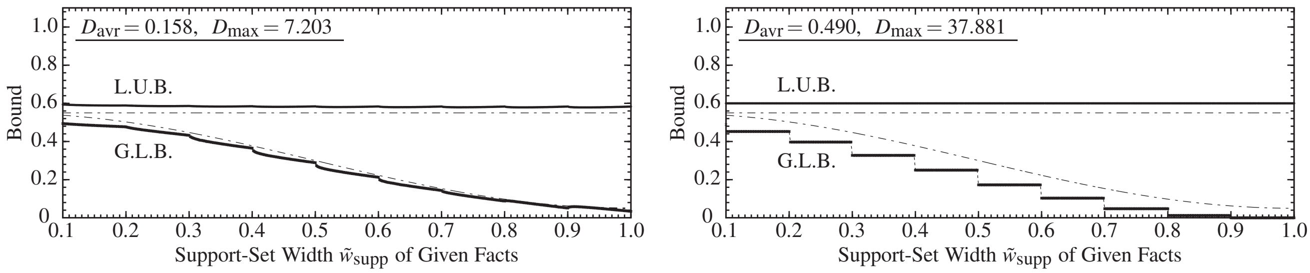

(a) $\alpha$-GEMII (proposed). (b) CRI-based parallel inference (conventional).

**Fig. 13.** Least upper and greatest lower bounds of $\tilde{Q}_{supp}$ where fuzzy sets in consequent parts are assigned by $q_{lnr}(x)$.




(a) $\alpha$-GEMII (proposed). (b) CRI-based parallel inference (conventional).

**Fig. 14.** Least upper and greatest lower bounds of $\tilde{Q}_{supp}$ where fuzzy sets in consequent parts are assigned by $q_{sgm}(x)$.




(a) $\alpha$-GEMII (proposed). (b) CRI-based parallel inference (conventional).

**Fig. 15.** Least upper and greatest lower bounds of $\tilde{Q}_{supp}$ where fuzzy sets in consequent parts are assigned by $q_{cos}(x)$.

the saturation of the least upper bounds of $\tilde{Q}_q$. **Fig. 15** shows that the greatest lower bounds of $\tilde{Q}_{supp}$, deduced by $\alpha$-GEMII, deviate in quite a narrow range and then $\alpha$-GEMII gives much smaller value of of $D_{max}$ than CRI-based parallel inference. In the range upper around 0.7 of $\tilde{w}_{supp}$ shown in **Fig. 15**, $\alpha$-GEMII properly follows the saturation of the greatest lower bounds of $\tilde{Q}_q$.

## 6. Conclusion

This paper has discussed the suppression effect of $\alpha$-GEMII on consequence deviations and numerically evaluated its effectiveness. Conventional fuzzy inference based on CRI has problems in discontinuous deviations of the least upper and greatest lower bounds of consequences even when it models continuous input-output relation of a system and given facts change continuously. In contrast, $\alpha$-GEMII can suppress the deviations with their self-tuning schemes in deducing consequences.

In this paper, first, the consequence deviations in CRI-based parallel inference have been illustrated in order to clarify their characteristics. Then, it has been shown that the $\alpha$-GEMII schemes contribute to suppressing the consequence deviations. Especially, the automatic reduction of the deviations is effective, which is performed by the schemes originally developed for fuzzy constraint propagation in $\alpha$-GEMII. In simulations, $\alpha$-GEMII and CRI-based parallel inference were numerically evaluated on the deviations and compared with each other. Simulation results show that $\alpha$-GEMII is superior to CRI-based parallel inference in suppressing the consequence deviations.

More precise analysis, including mathematical analysis, on the consequence deviations is for further study. The authors will present the applications of $\alpha$-GEMII in the future, making effective use of the properties in suppressing the consequence deviations. Especially, they will apply $\alpha$-GEMII to nonlinear prediction filters for complex time series with fluctuations which do not always originate from correlation between time series data.

**Acknowledgements**

The authors would like to thank Mr. Takeshi Watahiki, Ibaraki University, Japan, for his technical support in simulations.

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**Name:**
Kiyohiko Uehara

**Affiliation:**
Department of Media and Telecommunications Engineering, College of Engineering, Graduate School of Science and Engineering, Ibaraki University

**Address:**
4-12-1 Nakanarusawa-cho, Hitachi 316-8511, Japan

**Brief Biographical History:**
1998 Dr. Eng. degree received from Tokyo Institute of Technology
1983-1989 R & D Center, Toshiba Corporation
1989-1992 ATR Optical and Raid Communications Research Laboratories
1992-1994 Kansai Research Laboratory, Toshiba Corporation
1994-2004 R & D Center, Toshiba Corporation
2004- Ibaraki University

**Main Works:**
- K. Uehara, T. Koyama, and K. Hirota, "Inference with Governing Schemes for Propagation of Fuzzy Convex Constraints Based on $\alpha$-Cuts," J. of Advanced Computational Intelligence and Intelligent Informatics, Vol.13, No.3, pp. 321-330, 2009.
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- K. Uehara, T. Koyama, and K. Hirota, "Inference Based on $\alpha$-Cut and Generalized Mean with Fuzzy Tautological Rules," J. of Advanced Computational Intelligence and Intelligent Informatics, Vol.14, No.1, pp. 76-88, 2010.

**Membership in Academic Societies:**
- The Institute of Electrical and Electronics Engineers (IEEE)
- The Institute of Electronics, Information and Communication Engineers, Japan (IEICE)
- Japan Society for Fuzzy Theory and Intelligent Informatics (SOFT)

**Name:**
Takumi Koyama

**Affiliation:**
Department of Media and Telecommunications Engineering, College of Engineering, Ibaraki University (at this work)

**Address:**
4-12-1 Nakanarusawa-cho, Hitachi 316-8511, Japan

**Brief Biographical History:**
2003-2007 Undergraduate Student, Ibaraki University (at this work)

**Main Works:**
- K. Uehara, T. Koyama, and K. Hirota, "Inference with Governing Schemes for Propagation of Fuzzy Convex Constraints Based on $\alpha$-Cuts," J. of Advanced Computational Intelligence and Intelligent Informatics, Vol.13, No.3, pp. 321-330, 2009.
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**Name:**
Kaoru Hirota

**Affiliation:**
Department of Computational Intelligence and Systems Science, Tokyo Institute of Technology

**Address:**
G3-49 4259 Nagatsuta, Midori-ku, Yokohama 226-8502, Japan

**Brief Biographical History:**
1979-1982 Sagami Institute of Technology
1982-1995 Hosei University
1995 Tokyo Institute of Technology

**Main Works:**
- J. Guerra, H. Nobuhara, and K. Hirota, "Fuzzy Configuration Space for Moving Obstacle Avoidance of Autonomous Mobile Robots," J. of Advanced Computational Intelligence and Intelligent Informatics, Vol.10, No.1, pp. 26-34, 2006.
- K. Hirota, "Fuzzy Controlled Stepping Motors and Fuzzy Relation based Image Compression/Reconstruction (Banquet Keynote Speech)," 40th of Fuzzy Pioneers (1965-2005) BISC Special Event in Honor of Prof. Lotfi A. Zadeh (BISCSE2005), (U.C.Berkeley, U.S.A.), pp. 223-224, 2005.

**Membership in Academic Societies:**
- International Fuzzy Systems Association (IFSA) Fellow, President
- IEEE CIS Distinguished Lecturer