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Paper:

Axiomatization of Shapley Values of Fagle and Kern Type on Set Systems

Aoi Honda and Yoshiaki Okazaki

Department of Systems Design and Informatics, Kyushu Institute of Technology,
Iizuka, Fukuoka 820-8502, Japan
E-mail: {aoi, okazaki}@ces.kyutech.ac.jp

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We propose axiomatizing a generalized Shapley value of games for potential application to games on set systems satisfying the condition of normality. This encompasses both the original Shapley value and Faigle and Kern's Shapley value, which is generalized for a cooperative game defined on a subcoalition.

Keywords: cooperative game, shapley value, Faigle and Kern's Shapley value, normal set system

1. Introduction

Let $X = \{1, 2, \ldots, n\}$ be a set of players, for which any subset of X is called a *coalition*. Games are nonadditive functions from the set of coalitions to $\mathbb{R}$. The Shapley value is one of the game's solutions, which is the evaluation of the contribution of each player or the distribution of the profit for each player, charactarized by natural and understandable axiomatization [1]. The Shapley value is applied to the game defined on 2^X. It cannot, however, be applied to more general games. Generalizations of the Shapley value have been proposed by several authors. Algaba and et al. proposed a generalization of the Shapley value for games on antimatroids and characterized by an axiomatization that contains axiomatization of the original Shapley value [2]. On the other hand, Faige and Kern proposed another generalization of the Shapley value, and it has applicability to the cooperative game [3]. The axiomatization of Faige and Kern's Shapley value is not known, however.

We have proposed an entropy of the capacity on lattices using maximal chain concept, and given its axiomatization [4, 5]. Using the same idea, we generalize Faige and Kern's Shapley value to a more general form that is applicable both to the cooperative game and more general games, such as multichoice games and bicapacity, and characterize it by axiomatization from a perspective differing from the axiomatization of the original Shapley value.

This paper is organized as follows. Section 2 discusses preliminaries on set systems and games, then defines Faigle and Kern's Shapley value. Section 3 axiomatizes it and Section 4 proves the main theorem. Section 5 shows how these results can be applied to games defined on lattices and gives an example of a multichoice game application. Section 6 presents conclusions.

2. Games on Normal Set Systems

Throughout this paper, we consider a finite universal set $X = \{1, 2, \ldots, n\}$, $n \geq 1$, and 2^X denotes the power set of X. Consider $\mathfrak{S}$ a subset of 2^X that contains $\emptyset$ and X. We call $(X, \mathfrak{S})$ – or simply $\mathfrak{S}$ if no confusion occurs – a *set system*.

Let $A, B \in \mathfrak{S}$. We say that A is *covered* by B, and write $A \prec B$ or $B \succ A$, if $A \subsetneq B$ and $A \subseteq C \subsetneq B$ together with $C \in \mathfrak{S}$ imply $C = A$.

Definition 1—maximal chain of set system: Let $\mathfrak{S}$ be a set system. We call $\mathscr{C}$ a *maximal chain* of $\mathfrak{S}$ if $\mathscr{C} = (C_0, C_1, \ldots, C_m)$ satisfies $\emptyset = C_0 \prec C_1 \prec \cdots \prec C_m = X, C_i \in \mathfrak{S}, i = 0, \ldots, m$.

We denote by $\Gamma_k(\mathfrak{S})$ the set of all k-length maximal chains of $\mathfrak{S}$ $1 \leq k \leq n$, where the length of $\mathscr{C} = (C_0, C_1, \ldots, C_m)$ is m.

Definition 2—totally ordered set system: We say that $(X, \mathfrak{S})$ is a *totally ordered set system* if for any $A, B \in \mathfrak{S}$, either $A \subseteq B$ or $A \supsetneq B$.

If $(X, \mathfrak{S})$ is a totally ordered set system, then $(X, \mathfrak{S})$ has only one maximal chain.

Definition 3—normal set system: We say that $(X, \mathfrak{S})$ is a normal set system if for any $A \in \mathfrak{S}$, there exists n-length maximal chain $\mathscr{C} \in \Gamma_n(\mathfrak{S})$ satisfying $A \in \mathscr{C}$.

Definition 4—game on a set system: Let $(X, \mathfrak{S})$ be a set system. A function $v : \mathfrak{S} \to \mathbb{R}$ is a *game on* $(X, \mathfrak{S})$ if it satisfies $v(\emptyset) = 0$ and $v(X) = 1$.

The classical cooperative game is defined on $(X, 2^X)$.

Definition 5—Shapley value of classical game [1]: Let v be a game on $(X, 2^X)$. The Shapley value of v, $\Phi(v) = (\phi_1(v), \ldots, \phi_n(v)) \in \mathbb{R}^n$ is defined by

$$\phi_i(v) := \sum_{A \subseteq N \setminus \{i\}} \gamma^n_{|A|}(v(A \cup \{i\}) - v(A)),\ i = 1, \ldots, n,$$

where $\gamma_k^n := (n-k-1)!\,k!/n!$.

Note that $\sum_{i=1}^n \phi_i(v) = v(X)$ holds. The fact that the Shapley value can be represented using the maximal chains is well known in game theory.

Proposition 6: Fix $i \in X$ arbitrarily. For any $\mathscr{C} \in \Gamma_n(2^X)$, there exists the unique $A^i_{\mathscr{C}} \in \mathscr{C}$ such that $\{i\} \notin A^i_{\mathscr{C}}$, $A^i_{\mathscr{C}} \cup \{i\} \in \mathscr{C}$ and

$$\phi_i(v) = \frac{1}{n!} \sum_{\mathscr{C} \in \Gamma_n(2^X)} (v(A^i_{\mathscr{C}} \cup \{i\}) - v(A^i_{\mathscr{C}}))$$

holds.

We give a proof of Proposition 6 for the sake of completeness.

Proof: $|\Gamma_n(2^X)| = n!$ holds. Fix $i \in X$ arbitarily. First, we show that for any $\mathscr{C} \in \Gamma_n(2^X)$, there exists a unique $A^i_{\mathscr{C}} \in \mathscr{C}$ such that $\{i\} \notin A^i_{\mathscr{C}}$ and $A^i_{\mathscr{C}} \cup \{i\} \in \mathscr{C}$. Fix a $\mathscr{C} = (C_0, C_1, \ldots, C_m) \in \Gamma_n(2^X)$. We have for $i = 1, \ldots, m$, $|C_i \setminus C_{i-1}| = 1$ so that $m = n$ holds. We have $C_i \setminus C_{i-1} \neq C_j \setminus C_{j-1}$ for $i < j$ because if $C_i \setminus C_{i-1} = C_j \setminus C_{j-1}$ then $C_j \supseteq C_i \setminus C_{i-1}$. Since $i < j$, $C_{j-1} \supseteq C_i$, therefore $C_{j-1} \supseteq C_i \setminus C_{i-1}$, which is a contradiction. Hence, for any $\{i\} \subseteq X$, there is an $A^i_{\mathscr{C}} \in \mathscr{C}$ that satisfies $\{i\} \notin A^i_{\mathscr{C}}$ and $A^i_{\mathscr{C}} \cup \{i\} \in \mathscr{C}$.

Next we show that for $A \subset N \setminus \{i\}$, the number of chains that include $A \cup \{i\}$ and A is $(n-|A|-1)!|A|!$ Fix $i \in X$ arbitrarily. The number of chains from $A \cup \{i\}$ to N is $(n-|A|-1)!$ and chains from $\emptyset$ to A is $|A|!$ Hence the number of chains that include $A \cup \{i\}$ and A is $|A|! \cdot (n-|A|-1)!$ Therefore

$$\frac{1}{n!} \sum_{\mathscr{C} \in \Gamma_n(2^X)} (v(A^i_{\mathscr{C}} \cup \{i\}) - v(A^i_{\mathscr{C}}))$$
$$= \frac{(n-|A|-1)! \cdot |A|!}{n!} \sum_{A \in N \setminus \{i\}} (v(A \cup \{i\}) - v(A)),$$

which completes the proof. ∎

We extend Faigle and Kern's Shapley value to our framework.

Definition 7—Shapley value of game on set system: Let v be a game on a normal set system $(X, \mathfrak{S})$. The Shapley value of v, $\Phi(v) = (\phi_1(v), \ldots, \phi_n(v)) \in \mathbb{R}^n$ is defined by

$$\textbf{(FK)} \quad \phi_i(v) := \frac{1}{|\Gamma_n(\mathfrak{S})|} \sum_{\mathscr{C} \in \Gamma_n(\mathfrak{S})} (v(A^i_{\mathscr{C}} \cup \{i\}) - v(A^i_{\mathscr{C}})),$$

where $A^i_{\mathscr{C}} := A \in \mathscr{C} \in \Gamma_n(\mathfrak{S})$ such that $\{i\} \notin A$ and $A \cup \{i\} \in \mathscr{C}$.

Note that for each $\mathscr{C} \in \Gamma_n(\mathfrak{S})$, there exists a unique $A^i_{\mathscr{C}} := A \in \mathscr{C} \in \Gamma_n(\mathfrak{S})$ for any $i \in X$ (See Prop. 6).

We discuss the domain of Φ. Let v be a game on $(X, \mathfrak{S})$. We call $(X, \mathfrak{S}, v)$ a game space. Let Σ_n be the set of all normal set system of $X := \{1, 2, \ldots, n\}$ and let $\Delta_{\mathfrak{S}}$ be the set of all game space defined on normal set systems $(X, \mathfrak{S})$. The domain of Φ is $\Delta := \bigcup_{n=1}^{\infty} \bigcup_{\mathfrak{S} \in \Sigma_n} \Delta_{\mathfrak{S}}$, and Φ is a function defined on Δ to $\mathbb{R}^n$. We denote simply $\Phi(v)$ instead of $\Phi(X, \mathfrak{S}, v)$ so long as no confusion occurs.

We introduce further concepts about games, which will be useful for stating axioms.

Definition 8—dual game: Let v be a game on $(X, \mathfrak{S})$. The *dual game* of v is defined on $\mathfrak{S}^d := \{A^c \in 2^X \mid A \in \mathfrak{S}\}$ by $v^d(A) := 1 - v(A^c)$ for any $A \in \mathfrak{S}^d$, where $A^c := X \setminus A$.

Definition 9—permutation of v: Let π be a permutation on X. The *permutation* of v by π is defined on $\pi(\mathfrak{S}) := \{\pi(A) \in 2^X \mid A \in \mathfrak{S}\}$ by $\pi \circ v(A) := v(\pi^{-1}(A))$.

Consider a chain of length 2 as a set system, denoted by $\mathbf{2}$ (e.g., $\{\emptyset, \{1\}, \{1,2\}\}$), and a game $v^{\mathbf{2}}$ on it. We denote by the triplet $(0, s, t)$ the values of $v^{\mathbf{2}}$ along the chain and we assume $\mathbf{2} := \{\emptyset, \{1\}, \{1,2\}\}$ unless otherwise noted.

Definition 10—embedding of $v^{\mathbf{2}}$: Let v be a game on a totally ordered normal set system $(X, \mathfrak{S})$, where $\mathfrak{S} := \{C_0, \ldots, C_n\}$ such that $C_{i-1} \prec C_i$, $i = 1, \ldots, n$, and let $v^{\mathbf{2}} := (0, s, 1)$ be a game on $\mathbf{2}$. For $C_k \in \mathfrak{S}$, v^{C_k} is called the *embedding* of $v^{\mathbf{2}}$ into v at C_k and defined on the totally ordered normal set system $(X^{C_k}, \mathfrak{S}^{C_k})$ by

$$v^{C_k}(A) := \begin{cases} v(A), & \text{if } A = C_j, j < k, \\ v(C_{k-1}) + s \cdot \big(v(C_k) - v(C_{k-1})\big), & \text{if } A = C'_k, \\ v(C_{j-1}), & \text{if } A = C'_j, j > k, \end{cases} \quad (1)$$

where $\{i_k\} := C_k \setminus C_{k-1}, i'_k \neq i''_k$, $(X \setminus \{i_k\}) \cap \{i'_k, i''_k\} = \emptyset, X^{C_k} := (X \setminus \{i_k\}) \cup \{i'_k, i''_k\}, C'_k := (C_k \setminus \{i_k\}) \cup \{i'_k\}, C'_j := (C_{j-1} \setminus \{i_k\}) \cup \{i'_k, i''_k\}$ for $j > k$, and $\mathfrak{S}^{C_k} := \{C_0, \ldots, C_{k-1}, C'_k, C'_{k+1}, \ldots, C'_{n+1}\}$.

Note that more properly, the dual game of v is the dual game space of the game space $(X, \mathfrak{S}, v)$ that is defined by $(X, \mathfrak{S}, v)^d := (X, \mathfrak{S}^d, v^d)$, the permutation of v is the permutation of the game space $(X, \mathfrak{S}, v)$ that is defined by $(X, \mathfrak{S}, v)^{\pi} := (X, \pi(\mathfrak{S}), \pi \circ v)$, and the embedding of $(0, s, 1)$ into v is the embedding of the game space $(\{1,2\}, \mathbf{2}, (0, s, 1))$ into the game space $(X, \mathfrak{S}, v)$, and it is defined by $(X, \mathfrak{S}, v)^{C_k} := (X^{C_k}, \mathfrak{S}^{C_k}, v^{C_k})$.

3. Axiomatization of the Generalized Shapley Value

We introduce six axioms for our proposal Shapley value.

A1 (*continuity*). *For any game $(0, s, t)$ on $\mathbf{2}$, the function $\phi_1(0, s, t)$ is continuous for s on $\mathbb{R}$.*

A2 (*efficiency*). *For any game $(0, s, t)$ on $\mathbf{2}$, $\phi_1(0, s, t) + \phi_2(0, s, t) = v(X) = t$.*

A3 (*dual invariance*). *For any $(0, s, t)$, $\Phi(0, s, t) = \Phi(0, s, t)^d$ holds.*

A4 (*embedding efficiency*). *Let $(X, \mathfrak{S})$ be a totally ordered set system and let $\mathfrak{S} := \{C_0, \ldots, C_n\}, C_{i-1} \prec C_i, i =$*

$1,\ldots,n$. *For any* v *on* $(X,\mathfrak{S})$*, any* $(0,s,1)$ *and for any* $C_k \in \mathfrak{S}$*,* $\phi_i(v^{C_k}) = \phi_i(v)$ *for any* $i \neq i'_k, i''_k$*,* $\phi_{i'_k}(v^{C_k}) = \phi_{i_k}(v)\cdot\phi_1(0,s,1)$ *and* $\phi_{i''_k}(v^{C_k}) = \phi_{i_k}(v)\cdot\phi_2(0,s,1)$ *hold, where* $\{i_k\} := C_k \setminus C_{k-1} = \{i'_k, i''_k\}$.

A5 (*convexity*). *Let* $(X,\mathfrak{S})$*,* $(X,\mathfrak{S}_1)$ *and* $(X,\mathfrak{S}_2)$ *be normal set systems satisfying* $\Gamma_n(\mathfrak{S}_1)\cup\Gamma_n(\mathfrak{S}_2) = \Gamma_n(\mathfrak{S})$ *and* $\Gamma_n(\mathfrak{S}_1)\cap\Gamma_n(\mathfrak{S}_2) = \emptyset$ *and let* v *be a game on* $\mathfrak{S}$*. There exist* $\alpha \in]0,1[$ *such that for every game* v *on* $\mathfrak{S}$ *and for every* $i \in X$*, it holds that* $\phi_i(v) = \alpha\phi_i(v|_{\mathfrak{S}_1}) + (1-\alpha)\phi_i(v|_{\mathfrak{S}_2})$.

A6 (*permutation invariance*). *Let* v *be a game on* $(X,\mathfrak{S})$*. For any permutaion* π *on* X *satisfying* $\pi(\mathfrak{S}) = \mathfrak{S}$*,* $\phi_i(v) = \phi_{\pi(i)}(\pi\circ v), i = 1,\ldots,n$ *holds.*

We obtain the following theorem:

Theorem 11: Let v be a game on a normal set system $(X,\mathfrak{S})$. (FK) holds if and only if A1, A2, A3, A4, A5 and A6 hold.

We treat games defined on normal set systems. If the underlying space is not normal, Definition 7 cannot be applied to such games because $\phi_i(v)$ is calculated as an average of i's contributions $v(A\cup\{i\}) - v(A)$. For instance, let $X := \{1,2,3\}$. If v is defined on $\{\emptyset,\{1\},\{1,2,3\}\}$ which is not normal, we cannot know contributions of each single $\{2\}$ and $\{3\}$.

We discuss the above axioms in detail below.

3.1. Continuity

More generally, for any game on any normal set systems $(X,\mathfrak{S})$, $\phi_i(v)$ is continuous for v.

3.2. Efficiency

More generally, for any game on a normal set system, $\sum_i \phi_i(v) = v(X)$ holds.

Proposition 12: For any game on a normal set system $(X,\mathfrak{S})$, it holds that $\sum_{i=1}^n \phi_i(v) = v(X)$.

Proof: Let v be a game on $(X,\mathfrak{S})$. We then have

$$\begin{aligned}\sum_{i=1}^n \phi_i(v) &= \sum_{i=1}^n \left(\frac{1}{|\Gamma_n(\mathfrak{S})|}\sum_{\mathscr{C}\in\Gamma_n(\mathfrak{S})}(v(A\cup\{i\}) - v(A))\right)\\ &= \frac{1}{|\Gamma_n(\mathfrak{S})|}\sum_{\mathscr{C}\in\Gamma_n(\mathfrak{S})}\sum_{i=1}^n(v(A\cup\{i\}) - v(A))\\ &= \frac{1}{|\Gamma_n(\mathfrak{S})|}\sum_{\mathscr{C}\in\Gamma_n(\mathfrak{S})}(v(X) - v(\emptyset)) = v(X).\end{aligned}$$

■

3.3. Dual invariance

More generally, for any games on any normal set systems $(X,\mathfrak{S})$, $\Phi(v)$ is dual invariant.

Proposition 13: For any games v on a normal set system $(X,\mathfrak{S})$, $\Phi(v^d) = \Phi(v)$.

Proof: Let v be a game on $\mathfrak{S}$. For any $A\in\mathfrak{S}$, $(A^c)^c = A$, hence the dual mapping is a bijection from $\mathfrak{S}$ to $\mathfrak{S}^d$. Then, $\mathscr{C} := (C_0,\ldots,C_n)\in\Gamma_n(\mathfrak{S})$ if and only if $\mathscr{C}^d := (C_n^c,\ldots,C_0^c)\in\Gamma_n(\mathfrak{S}^d)$, since $C_j \prec C_{j+1}$ implies $C_j^c \succ C_{j+1}^c$. Hence $|\Gamma_n(\mathfrak{S})| = |\Gamma_n(\mathfrak{S}^d)|$. Therefore

$$\begin{aligned}\phi_i(v^d) &= \frac{1}{|\Gamma_n(\mathfrak{S}^d)|}\sum_{\mathscr{C}\in\Gamma_n(\mathfrak{S}^d)}\left(v^d(A^i_{\mathscr{C}}\cup\{i\}) - v^d(A^i_{\mathscr{C}})\right)\\ &= \frac{1}{|\Gamma_n(\mathfrak{S}^d)|}\sum_{\mathscr{C}\in\Gamma_n(\mathfrak{S}^d)}\Big((1 - v^d(A^i_{\mathscr{C}}))\\ &\qquad - (1 - v^d(A^i_{\mathscr{C}}\cup\{i\}))\Big)\\ &= \frac{1}{|\Gamma_n(\mathfrak{S})|}\sum_{\mathscr{C}\in\Gamma_n(\mathfrak{S})}(v(B\cup\{i\}) - v(B)) = \phi_i(v),\end{aligned}$$

where $A^i_{\mathscr{C}} := A\in\mathscr{C}\in\Gamma_n(\mathfrak{S})$ such that $\{i\}\notin A$ and $A\cup\{i\}\in\mathscr{C}$, and $B := X\setminus(A^i_{\mathscr{C}}\cup\{i\})$. ■

3.4. Embedding Efficiency

Let v be a game on a totally ordered set system $\mathfrak{S} := \{C_0,\ldots,C_n\}$ such that $C_{i-1}\prec C_i$, $i = 1,\ldots,n$, The embedding at C_k into v by $(0,u,1)$ means that $i_k := C_k\setminus C_{k-1}$ is split into $\{i'_k, i''_k\}$ whose contributions are $\phi_1(0,s,1)$ and $\phi_2(0,s,1)$. A3 implies $\phi_{i'_k}(v^{C_k}) + \phi_{i''_k}(v^{C_k}) = \phi_{i_k}(v)$, so that A3 is a natural axiom in the meaning of the contributions of players i'_k and i''_k.

4. Proof of Theorem

Before the proof of Theorem 11, we show a lemma needed later.

Lemma 14: [6] Let $f(x)$ be a continuous function on $\mathbb{R}$.
(i) For any x,y, $f(x+y) = f(x)+f(y)$ holds if and only if $f(x) = \alpha x, \alpha\in\mathbb{R}$.
(ii) For any x,y, $f(x+y) = f(x)f(y)$ holds if and only if $f(x) = e^{\alpha x}, \alpha\in\mathbb{R}$, or constant valued $f(x) = 0$.
(iii) For any x,y, $f(xy) = f(x)f(y)$ holds if and only if $f(x) = |x|^\alpha$ or $f(x) = \mathrm{sign}(x)|x|^\alpha, \alpha\geq 0$.

Lemma 15: For any normal set systems $(X,\mathfrak{S})$ and $(X,\mathfrak{S}')$ satisfying $\mathfrak{S}_1 \subsetneq \mathfrak{S}$, there exist $\mathfrak{S}_1$ and $\mathfrak{S}_2$ such that $\Gamma_n(\mathfrak{S}_1)\cup\Gamma_n(\mathfrak{S}_2) = \Gamma_n(\mathfrak{S})$, $\Gamma_n(\mathfrak{S}_1)\cap\Gamma_n(\mathfrak{S}_2) = \emptyset$, $\mathfrak{S}_1\subsetneq\mathfrak{S}'$ and $\mathfrak{S}_2\subsetneq\mathfrak{S}$.

Proof: Choose one element as A from $\mathfrak{S}\setminus\mathfrak{S}'$, and put $\mathfrak{S}_1 := \{B\in\mathfrak{S}\mid B\subseteq A$ or $B\supsetneq A\}$ and $\mathfrak{S}_2 := \{B\in\mathfrak{S}\mid B\subseteq A'$ or $B\supsetneq A'$, where $A'\in\mathfrak{S}\setminus A$ satisfying $|A'| = |A|\}$. Then $\Gamma_n(\mathfrak{S}_1)$ contains all elements that contain A and $\Gamma_n(\mathfrak{S}_2)$ contains all elements that do not contain A of $\Gamma_n(\mathfrak{S})$. ■

Lemma 16: For any normal set systems $(X,\mathfrak{S})$ satisfying $|\Gamma_n(\mathfrak{S})| \geq 2$, there exist $\mathfrak{S}_1$ and $\mathfrak{S}_2$ such that $\Gamma_n(\mathfrak{S}_1) \cup \Gamma_n(\mathfrak{S}_2) = \Gamma_n(\mathfrak{S})$ and $\Gamma_n(\mathfrak{S}_1) \cap \Gamma_n(\mathfrak{S}_2) = \emptyset$.

Proof: Choose one element as A from $\mathfrak{S}$ that satisfies that there exists $A' \in \mathfrak{S} \setminus A$ such that $|A'| = |A|$ and put $\mathfrak{S}_1 := \{B \in \mathfrak{S} \mid B \subseteq A \text{ or } B \supsetneq A\}$ and $\mathfrak{S}_2 := \{B \in \mathfrak{S} \mid \exists A' \in \mathfrak{S} \setminus A \text{ such that } B \subseteq A' \text{ or } B \supsetneq A'\}$. Then $\Gamma_n(\mathfrak{S}_1)$ contains all elements that contain A and $\Gamma_n(\mathfrak{S}_2)$ contains all elements which do not contain A of $\Gamma_n(\mathfrak{S})$. ■

We now show the proof of Theorem 11.

Proof: **(necessity)** $\phi_1(0,s,t) = s$ clearly satisfies A1, and by Propositions 12 and 13, (FK) satisfies A2 and A3, respectively.

We show that (FK) satisfies A4. Let v be a game on $\mathfrak{S} := \{C_0, \dots, C_n\}$ such that $C_{i-1} \prec C_i$, $i = 1, \dots, n$. The game v^{C_k} that is an embedding of $(0,s,1)$ into v at C_k is defined on $\mathfrak{S}^{C_k} = \{C_0, \dots, C_{k-1}, C_{k'}, C_k, C_{k+1}, \dots, C_n\}$ such that $C_{k'} \prec C_k \prec C_{k+1}$. We then have

$$\phi_i(v^{C_k}) = \phi_i(v), \quad i \neq i'_k, i''_k,$$

$$\begin{aligned}\phi_{i'_k}(v^{C_k}) &= v^{C_k}(C'_k) - v^{C_k}(C_{k-1}) \\ &= (v(C_{k-1}) + u \cdot (v(C_k) - v(C_{k-1}))) - v(C_{k-1}) \\ &= (v(C_k) - v(C_{k-1})) \cdot u = \phi_{i_k}(v) \cdot \phi_1(0,s,1)\end{aligned}$$

and

$$\begin{aligned}\phi_{i''_k}(v^{C_k}) &= v^{C_k}(C'_{k+1}) - v^{C_k}(C'_k) \\ &= v(C_k) - (v(C_{k-1}) + u \cdot (v(C_k) - v(C_{k-1}))) \\ &= (v(C_k) - v(C_{k-1}))(1-s) \\ &= \phi_{i_k}(v) \cdot \phi_2(0,s,1),\end{aligned}$$

that imply A4.

We show that (FK) satisfies A5. Let v be a game on $\mathfrak{S}$. Then we have

$$\begin{aligned}\phi_i(v) &= \frac{1}{|\Gamma_n(\mathfrak{S})|} \sum_{\mathscr{C} \in \Gamma_n(\mathfrak{S})} (v(A^i_{\mathscr{C}} \cup \{i\}) - v(A^i_{\mathscr{C}})) \\ &= \frac{1}{|\Gamma_n(\mathfrak{S})|} \left(\sum_{\mathscr{C} \in \Gamma_n(\mathfrak{S}_1)} + \sum_{\mathscr{C} \in \Gamma_n(\mathfrak{S}_2)} \right) (v(A^i_{\mathscr{C}} \cup \{i\}) - v(A^i_{\mathscr{C}})) \\ &= \frac{|\Gamma_n(\mathfrak{S}_1)|}{|\Gamma_n(\mathfrak{S})|} \frac{1}{|\Gamma_n(\mathfrak{S}_1)|} \sum_{\mathscr{C} \in \Gamma_n(\mathfrak{S}_1)} (v|_{\mathfrak{S}_1}(A^i_{\mathscr{C}} \cup \{i\}) - v|_{\mathfrak{S}_1}(A^i_{\mathscr{C}})) \\ &\quad + \frac{|\Gamma_n(\mathfrak{S}_2)|}{|\Gamma_n(\mathfrak{S})|} \frac{1}{|\Gamma_n(\mathfrak{S}_2)|} \sum_{\mathscr{C} \in \Gamma_n(\mathfrak{S}_2)} (v|_{\mathfrak{S}_2}(A^i_{\mathscr{C}} \cup \{i\}) - v|_{\mathfrak{S}_2}(A^i_{\mathscr{C}})) \\ &= \frac{|\Gamma_n(\mathfrak{S}_1)|}{|\Gamma_n(\mathfrak{S})|} \phi_i(v|_{\mathfrak{S}_1}) + \frac{|\Gamma_n(\mathfrak{S}_2)|}{|\Gamma_n(\mathfrak{S})|} \phi_i(v|_{\mathfrak{S}_2})\end{aligned}$$

and

$$\frac{|\Gamma_n(\mathfrak{S}_1)|}{|\Gamma_n(\mathfrak{S})|} + \frac{|\Gamma_n(\mathfrak{S}_2)|}{|\Gamma_n(\mathfrak{S})|} = 1.$$

(sufficiency) First, we show that for any capacity on a totally ordered normal set system **2**, A1, A2 and A3 implies (FK). Without loss of generality, we may assume $t = 1$. Put $f(x) := \phi_1(0,x,1)$, then we have $\phi_1(0,x,1) + \phi_2(0,x,1) = 1$, so that $\phi_2(0,x,1) = 1 - f(x)$ by A2 and

$$f(x) = 1 - f(1-x) \quad \dots \dots \quad (2)$$

by A3. Assume that $v^{\mathbf{3}}$ is a capacity on $(X := \{1,2,3\}, \{\emptyset, \{1\}, \{1,2\}, X\})$ and $v^{\mathbf{3}}(1) := a, v^{\mathbf{3}}(\{1,2\}) := a+b, v^{\mathbf{3}}(X) := a+b+c = 1, a,c \geq 0, b > 0$. We can regard $v^{\mathbf{3}}$ as the embedding of $v^{\mathbf{2}} = (0, b/(1-a), 1)$ into $v = (0,a,1)$ on $(\emptyset, \{1\}, X)$ at X. Then, by A4, we get

$$\phi_1(v^{\mathbf{3}}) = \phi_1(0,a,1) = f(a). \quad \dots \dots \quad (3)$$

Similarly, we can also regard $v^{\mathbf{3}}$ as the embedding of with $v^{\mathbf{2}} = (0, a/(a+b), 1)$ into $v = (0, a+b, 1)$ on $(\emptyset, \{1,2\}, X)$ at $\{1,2\}$. Then we obtain by A4,

$$\begin{aligned}\phi_1(v^{\mathbf{3}}) &= \phi_1(0,a+b,1)\phi_1(0,a/(a+b),1) \\ &= f(a+b)f(a/(a+b)).\end{aligned} \quad (4)$$

Eqs. (3) and (4) yield

$$f(a) = f(a+b)f(a/(a+b)). \quad \dots \dots \quad (5)$$

Putting $x := a+b$ and $y := a/(a+b)$, we have $f(xy) = f(x)f(y)$, so that by Lemma 14 $f(x) = x^{\alpha}, \alpha \in \mathbb{R}$. Putting $x = 1/2$ in (2), we obtain $f(x) = x$. Hence we have $\phi_1(0,x,1) = f(x) = x$ and $\phi_2(0,x,1) = 1-x$, so that for any $(0,x,1)$, therefore for any $(0,s,t)$, (FK) holds.

Assume that (FK) holds for a game v on the totally ordered normal set system $\mathfrak{S} = \{C_0, C_1, \dots, C_n\}$, $C_{i-1} \prec C_i$, $i = 1, \dots, n$ Then by A3, for v^{C_k}, we have $\phi_i(v^{C_k}) = \phi_i(v)$ for any $i \neq i'_k, i''_k$,

$$\begin{aligned}\phi_{i'_k}(v^{C_k}) &= \phi_{i_k}(v) \cdot \phi_1(0,u,1) \\ &= \big(v(C_k) - v(C_{k-1})\big) \cdot u = v^{C_k}(C'_k) - v^{C_k}(C_{k-1})\end{aligned}$$

and

$$\begin{aligned}&\phi_{i''_k}(v^{C_k}) \\ &= \phi_{i_k}(v) \cdot \phi_1(0,u,1) \\ &= \big(v(C_k) - v(C_{k-1})\big) \cdot (1-u) = v^{C_k}(C'_{k+1}) - v^{C_k}(C'_k),\end{aligned}$$

where $\{i_k\} := C_k \setminus C_{k-1} = \{i'_k, i''_k\}$ by (1) and A3, which means (FK) holds for v^{C_k}, so that (FK) holds for any games on totally ordered normal set systems.

Next, we show that for normal set system that is not totally ordered, (FK) also holds. Fix a normal set system $\mathfrak{S} \subseteq 2^X$. By Lemma 15, there exist $\mathfrak{S}_1, \dots, \mathfrak{S}_k$ and $\alpha, \alpha_1, \dots, \alpha_k$, and we have

$$\begin{aligned}&\Gamma_n(2^X) = \Gamma_n(\mathfrak{S}) \cup \Gamma_n(\mathfrak{S}_1) \cup \dots \cup \Gamma_n(\mathfrak{S}_k), \\ &\Gamma_n(\mathfrak{S}) \cap \Gamma_n(\mathfrak{S}_1) \cap \dots \cap \Gamma_n(\mathfrak{S}_k) = \emptyset, \\ &\phi_i(v) = \alpha\phi_i(v|_{\mathfrak{S}}) + \alpha_1\phi_i(v|_{\mathfrak{S}_1}) + \dots + \alpha_k\phi_i(v|_{\mathfrak{S}_k})\end{aligned} \quad (6)$$

for $i = 1, \ldots, n$ and $\alpha + \alpha_1 + \cdots + \alpha_k = 1$. Applying Lemma 16 successively, we obtain totally ordered normal set systems $\mathfrak{S}^1, \ldots, \mathfrak{S}^\ell, \mathfrak{S}_1^1, \ldots \mathfrak{S}_1^{\ell(1)}, \ldots, \mathfrak{S}_k^1, \ldots \mathfrak{S}_k^{\ell(k)}$ and it holds that

$$\begin{aligned}\phi_i(v) &= \alpha^1 \phi_i(v|_{\mathfrak{S}^1}) + \cdots + \alpha^l \phi_i(v|_{\mathfrak{S}^\ell}) \\ &\quad + \alpha_1^1 \phi_i(v|_{\mathfrak{S}_1^1}) + \cdots + \alpha_1^{\ell(1)} \phi_i(v|_{\mathfrak{S}_{\ell(1)}^{\ell(1)}}) \\ &\quad + \cdots \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (7) \\ &\quad + \alpha_k^1 \phi_i(v|_{\mathfrak{S}_k^1}) + \cdots + \alpha_k^{\ell(k)} \phi_i(v|_{\mathfrak{S}_k^{\ell(k)}}) \\ &=: \beta_{\mathscr{C}_1} \phi_i(v|_{\mathscr{C}_1}) + \cdots + \beta_{\mathscr{C}_{n!}} \phi_i(v|_{\mathscr{C}_{n!}}),\end{aligned}$$

for $i = 1, \ldots, n$, $\alpha^1 + \cdots + \alpha^\ell = \alpha, \alpha_1^1 + \cdots + \alpha_1^{\ell(1)} = \alpha_1$, $\ldots, \alpha_k^1 + \cdots + \alpha_k^{\ell(k)} = \alpha_k$ and $\beta_{\mathscr{C}_1} + \ldots + \beta_{\mathscr{C}_{n!}} = 1$. Define a game v_j, $j = 1$, on 2^X by

$$v_j(A) := \begin{cases} 0, & |A| < j \\ 1, & |A| \geq j \end{cases}$$

Then we have

$$\begin{aligned}\phi_i(v_j) &= \sum_{\substack{\mathscr{C} \in \Gamma_n(2^X), \mathscr{C} \ni A, A \setminus \{i\}, \\ A \ni \{i\}, |A| = j}} \beta_{\mathscr{C}} \phi_i(v|_{\mathscr{C}}) \\ &= \sum_{\substack{\mathscr{C} \in \Gamma_n(2^X), \mathscr{C} \ni A, A \setminus \{i\}, \\ A \ni \{i\}, |A| = j,}} \beta_{\mathscr{C}}.\end{aligned}$$

By A6, for any permutation π on X,

$$\begin{aligned}\phi_i(v_j) &= \phi_{\pi(i)}(\pi \circ v_j) \\ &= \phi_{\pi^{-1}(i)}(v_j)\end{aligned}$$

holds, so that we have

$$\sum_{\substack{\mathscr{C} \in \Gamma_n(2^X), \mathscr{C} \ni A, A \setminus \{1\}, \\ A \ni \{1\}, |A| = j}} \beta_{\mathscr{C}} = \cdots = \sum_{\substack{\mathscr{C} \in \Gamma_n(2^X), \mathscr{C} \ni A, A \setminus \{n\}, \\ A \ni \{n\}, |A| = j}} \beta_{\mathscr{C}}$$

for $i = 1, \ldots, n$ and $j = 1, \ldots, n$, which yields

$$\beta_{\mathscr{C}_1} = \cdots = \beta_{\mathscr{C}_{n!}} = \frac{1}{n!}. \quad \ldots\ldots\ldots\ldots \quad (8)$$

Substituting Eq. (8) for Eqs. (6) and (7), we obtain

$$\begin{aligned}\alpha \phi_i(v|_{\mathfrak{S}}) &= \beta_{\mathfrak{S}_1} \phi_i(v|_{\mathfrak{S}_1}) + \cdots \beta_{\mathfrak{S}_\ell} \phi_i(v|_{\mathfrak{S}_\ell}) \\ &= \frac{1}{n!} \left(\phi_i(v|_{\mathfrak{S}_1}) + \cdots \phi_i(v|_{\mathfrak{S}_\ell}) \right),\end{aligned}$$

so that

$$\begin{aligned}&\phi_i(v|_{\mathfrak{S}}) \\ &= \frac{1}{|\Gamma_n(\mathfrak{S})|} \sum_{\mathscr{C} \in \Gamma_n(\mathfrak{S})} \phi_i(v|_{\mathscr{C}}) \\ &= \frac{1}{|\Gamma_n(\mathfrak{S})|} \sum_{\mathscr{C} \in \Gamma_n(\mathfrak{S})} \left(v|_{\mathfrak{S}}(A_{\mathscr{C}}^i \cup \{i\}) - v|_{\mathfrak{S}}(A_{\mathscr{C}}^i) \right).\end{aligned}$$

■

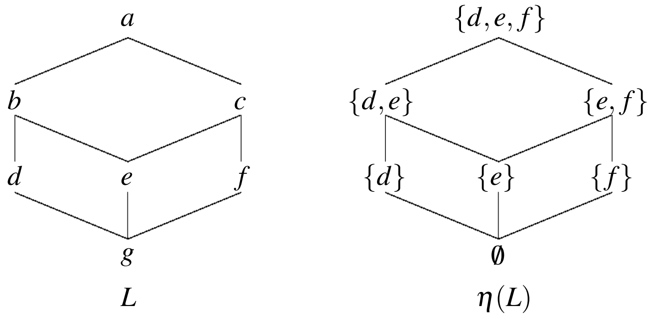

Fig. 1. Translation of lattice

5. Application to Game on Lattice

The *lattice* $(L, \leq)$ is a partially ordered set such that for any pair $x, y \in L$, there exist a least upper bound $x \vee y$ (supremum) and a greatest lower bound $x \wedge y$ (infimum) in L. Consequently, for finite lattices, there always exist a greatest element (supremum of all elements) and a least element (infimum of all elements), denoted by $\top, \bot$ (see [7]). Our approach may have applicability to games defined on lattices that satisfy normality by translation from lattices to set systems (cf. [4]).

Evidently a set system is not necessarily a lattice. Moreover, a normal set system is not necessarily a lattice. Indeed, take $X = \{1,2,3,4\}$ and $\mathfrak{S} := \{\emptyset, \{1\}, \{3\}, \{1,2\}, \{2,3\}, \{1,4\}, \{3,4\}, \{1,2,3\}, \{1,3,4\}, X\}$. Then, $\{1\}$ and $\{3\}$ have no supremum.

We can translate lattices to set systems that consisted by its join-irreducible elements.

Definition 17—join-irreducible element: An element $x \in (L, \leq)$ is *join-irreducible* if for all $a, b \in L$, $x \neq \bot$ and $x = a \vee b$ implies $x = a$ or $x = b$.
We denote by $\mathscr{J}(L)$ the set of all join-irreducible elements of L. Similarly, meet-irreducible elements are defined by replacing $\vee$ by $\wedge$ in the above definition. The set of all meet-irreducible elements is denoted by $\mathscr{M}(L)$.

The mapping η for any $a \in L$, defined by

$$\eta(a) := \{x \in \mathscr{J}(L) \mid x \leq a\}$$

is a lattice-isomorphism of L onto $\eta(L) := \{\eta(a) \mid a \in L\}$, that is, $(L, \leq) \cong (\eta(L), \subseteq)$.

5.1. Example – Multichoice Game

Let $N := \{0, 1, \ldots .n\}$ be a set of players, and let $L := L_1 \times \cdots \times L_n$, where $(L_i, \leq_i)$ is a totally ordered set $L_i = \{0, 1, \ldots, \ell_i\}$ such that $0 \leq_i 1 \leq_i \cdots \leq_i \ell_i$. Each L_i is the set of choices of player i. $(L, \leq)$ is a normal lattice. For any $(a_1, a_2, \ldots, a_n), (b_1, b_2, \ldots, b_n) \in L$, $(a_1, a_2, \ldots, a_n) \leq (b_1, b_2, \ldots, b_n)$ iff $a_i \leq_i b_i$ for all $i = 1, \ldots, n$. We have

$$\mathscr{J}(L) = \{(0, \ldots, 0, a_i, 0, \ldots, 0) \mid a_i \in \mathscr{J}(L_i) = L_i \setminus \{0\}\}$$

and $|\mathscr{J}(L)| = \sum_{i=1}^{n} \ell_i$. In this case, applying Definition 7, we obtain

$$\phi_i^j(v) = \phi_{(0,\ldots,0,a_i=j>0,0,\ldots,0)}(v)$$
$$= \sum_{a\in L/L_i} \xi_i^{(a,j)} \left(v(a,j) - v(a,j-1)\right),$$

where $L/L_i := L_1 \times \cdots \times L_{i-1} \times L_{i+1} \times \cdots \times L_n$, $(a,a_i) := (a_1,\ldots,a_{i-1},a_i,a_{i+1},\ldots,a_n) \in L$ such that $a \in L/L_i$ and $a_i \in L_i$, and

$$\xi_i^{(a,a_i)} := \left(\prod_{k=1}^{n} \binom{\ell_k}{a_k}\right) \cdot \binom{\sum_{k=1}^{n} \ell_k}{\sum_{k=1}^{n} a_k}^{-1} \cdot \frac{a_i}{\sum_{k=1}^{n} a_k}.$$

$\phi_i^j(v)$ represents the contribution of player i playing at level j compared to level $j-1$, where $j, j-1 \in \mathscr{J}(L_i) = L_i \setminus \{0\}$, hence player i's overall contribution is given by

$$\phi_i(v) = \sum_{j=1}^{\ell_i} \phi_i^j(v).$$

$\xi_i^{(a,a_i)}$ is the rate of the number of chains which contain (a,a_i) and (a,a_i-1). In fact,

$$|\{C \in \mathscr{C}(L) \mid C \ni (a,a_i),(a,a_i-1)\}|$$
$$= \frac{(\sum_{k=1}^{n} a_k - 1)!}{(\prod_{k=1}^{n}(a_k!))(a_i-1)!/(a_i!)} \cdot \frac{(\sum_{k=1}^{n}(\ell_k - a_k))!}{\prod_{k=1}^{n}((\ell_k - a_k)!)}$$

and $|\mathscr{C}(L)| = (\sum_{k=1}^{n} \ell_k)!/\prod_{k=1}^{n}(\ell_k!)$.

6. Conclusions

We have proposed axiomatizing the Shapley value of games defined on normal set systems that accords to Faigle and Kern's Shapley value for a cooperative game. Its definition, an average contribution of each player of all maximal chains, appears adequate and its axiomatization is natural and understandable. In considering the following example, let v be a game defined on $(X,\mathfrak{S})$ where $X = \{1,2,3\}, \mathfrak{S} = \{\emptyset,\{1\},\{2\},\{1,2\},\{1,2,3\}\}$ and $v(\emptyset) = 0, v(\{1\}) = 0.01, v(\{2\}) = 0, v(\{1,2\}) = 0.01, v(\{1,2,3\}) = 1$. Algaba et al.'s solution is $(\phi_1(v),\phi_2(v),\phi_3(v)) = (0.34,0.33,0.33)$, and our solution is $(\phi_1(v),\phi_2(v),\phi_3(v)) = (0.01,0,0.99)$.

In previous work, we discussed the entropy of capacities defined on a regular set system, not a normal set system [4, 5]. The definition of the regular set system is as follows:

Definition 18—regular set system: We say that $(X, \mathfrak{S})$ is a regular set system if all maximal chains of $\mathfrak{S}$ are n-length.

Our entropy is modifiable for applying capacities defined on normal set systems similar to the Shapley value. Its fourth axiom [5] is slightly modified to our form similar to A5 here.

References:

[1] L. Shapley, "A value for n-person games," In H. Kuhn and A. Tucker (Eds.), Contributions to the Theory of Games, Vol.2, 28, in Annals of Mathematics Studies, pp. 307-317, Princeton University Press, 1953.

[2] E. Algaba and J. M. Bilbao, R. van den Brink, and A. Jime'nez-Losada, "Aximatizations of the Shapley value for cooperative games on antimatroids," Math. Meth Oper. Res. 57, pp. 49-65, 2003.

[3] U. Faigle and W. Kern, "The Shapley value for cooperative games under precedence constraints," Int. J. of Game Theory, 21, pp. 249-266, 1992.

[4] A. Honda and M. Grabisch, "Entropy of capacities on lattices," Information Sciences, 176, pp. 3472-3489, 2006.

[5] A. Honda and M. Grabisch, "An axiomatization of entropy of capacities on set systems," European Journal of Operational Research, in press.

[6] J. Aczel, "Lectures on functional equations and their applications," Academic Press, 1966.

[7] B. A. Davey and H. A. Priestley, "Introduction to lattices and order," Cambridge University Press, 1990.

[8] A. Dukhovny, "General entropy of general measures," Int. J. Uncertain. Fuzziness Knowledge-Based Systems, 10, pp. 213-225, 2002.

[9] M. Grabisch, "An axiomatization of the Shapley value and interaction index for games on lattices," SCIS-ISIS 2004, 2nd Int. Conf. on Soft Computing and Intelligent Systems and 5th Int. Symp. on Advanced Intelligent Systems, Yokohama, Japan, 2004.

[10] C. R. Hsiao and T. E. S. Raghavan, "Shapley value for multichoice cooperative games I," Games and Economic Behavior, 5, pp. 240-256, 1993.

[11] I. Kojadinovic, J.-L. Marichal, and M. Roubens, "An axiomatic approach to the definition of the entropy of a discrete Choquet capacity," Information Sciences, 172, pp. 131-153, 2005.

[12] J.-L. Marichal and M. Roubens, "Entropy of discrete fuzzy measure," Int. J. Uncertain. Fuzziness Knowledge-Based Systems, 8, pp. 625-640, 2000.

[13] C. E. Shannon, "A mathematical theory of communication," Bell System Tech. Journ, 27, pp. 374-423, pp. 623-656, 1948.

Name:
Aoi Honda

Affiliation:
Assistant Professor, Department of Systems Design and Informatics, Faculty of Computer Science and Systems Engineering, Kyushu Institute of Technology

Address:
680-4,Kawazu, Iizuka, Fukuoka 820-8502 Japan

Brief Biographical History:
Assistant Professor, Department of Systems Design and Informatics, Faculty of Computer Science and Systems Engineering, Kyushu Institute of Technology
2004-2005 Visiting Scientist, Department of Computer science and Mathematics, University of Paris I , Pantheon-Sorbonne

Main Works:
• A. Honda and M. Grabisch, "Entropy of capacities on lattices and set systems," Information Sciences, Vol.176, pp. 3472-3489, 2006.

Membership in Academic Societies:
• Mathematical Society of Japan (MSJ)
•Japan Society for Fuzzy Theory and Informatics (JSFTI)
•The Institute of Electronics, Information and Communication Engineers (IEICE)
•Information Processing Society of Japan (IPSJ)

Name:
Yoshiaki Okazaki

Affiliation:
Professor, Department of Systems Design and Informatics, Faculty of Computer Science and Systems Engineering, Kyushu Institute of Technology

Address:
680-4,Kawazu, Iizuka, Fukuoka 820-8502 Japan

Brief Biographical History:
1989- Assistant Professor, Department of Systems Design and Informatics, Faculty of Computer Science and Systems Engineering, Kyushu Institute of Technology
1993- Professor, Department of Systems Design and Informatics, Faculty of Computer Science and Systems Engineering, Kyushu Institute of Technology

Main Works:
• A. Honda, Y. Okazaki, and H. Sato, "An Lp-function determines lp," Proc. of the Japan Academy, Vol.84, Series A Mathematical Science, pp. 39-41, Japan Academy, 2008.

Membership in Academic Societies:
• Mathematical Society of Japan (MSJ)
•Japan Society for Fuzzy Theory and Informatics (JSFTI)
•The Institute of Electronics, Information and Communication Engineers (IEICE)