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Paper:

Comparison of Fractional Robust- and Fixed Point Transformations- Based Adaptive Compensation of Dynamic Friction

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The features of fractional order robust and fixed-point transformation based adaptive controllers of a "Ball-Beam System" are compared to each other. The speciality of this task is that the position of the ball along the beam is indirectly controlled via directly controlling the other axis, the tilting angle of the beam. It is assumed that this tilting axle suffers from considerable dynamic friction mathematically approximated by the LuGre model. By neglecting the internal physics of the tilting drive this system can be modeled as a 4th order one because only the 4th time-derivative of the ball's position can directly be influenced by the tilting torque. The system also has saturation since the available acceleration of the ball is limited by the gravitation. It is shown that little reduction of the order of the differential equation controlling the decay of the error metrics in a Sliding Mode / Variable Structure controller considerably improves the robust controller. However, really precise solution can be obtained by the adaptive controller. These statements are illustrated and substantiated via simulation.

Keywords: robust control, variable structure / sliding mode control, adaptive control, fixed point transformations, iterative control

1. Introduction

The mathematical foundation of the modern *Soft Computing (SC)* techniques goes back to the middle of the 20th century, namely to the first rebuttal of David Hilbert's 13th conjecture [1] that was delivered by Arnold [2] and Kolmogorov [3] in 1957. Hilbert supposed that there exist such continuous multi-variable functions that cannot be decomposed as the finite superposition of continuous functions of less variables. Kolmogorov provided a constructive proof stating that arbitrary continuous function on a compact domain can be approximated with arbitrary accuracy by the composition of single-variable continuous functions. Though the construction of Kolmogorov's functions that are used in this theorem is difficult, his theorem later was found to be the mathematical basis of the present *SC* techniques.

From the late eighties several authors proved that different types of neural networks possessed the universal approximation property [4–7]. Similar results have been published from the early nineties in fuzzy theory claiming that different fuzzy reasoning methods are related to universal approximators, too [8–10].

In spite of these theoretically inspiring and promising conclusions, from the point of view of the practical applicability of these methods various theoretical doubts emerged. The most significant problem was, and remained important problem even in our days, the *"curse of dimensionality"* that means that the approximating models have exponential complexity in terms of the number of components i.e. the number of components grows exponentially as the approximation error tends to zero. If the number of the components is bounded, the resulting set of models is nowhere dense in the space of the approximated functions. These observations frequently were formulated in a negatory style, as e.g. in [11] stating that *"Sugeno controllers with a bounded number of rules are nowhere dense"*, and initiated various investigations on the nowhere denseness of certain fuzzy controllers containing prerestricted number of rules e.g. in [12–14].

In contrast to these observations *SC* techniques obtained very wide range of real practical applications. As examples implementation of backward identification methods [15], the control of a furnace testing various features of plastic threads [16, 17], sensor data fusion [18] can be mentioned. The methodology of the *SC* techniques, partly concerning control applications, had fast theoretical development in recent years, too. Various operators concerning the operation of the fuzzy inference processes were investigated in [19, 20], minimum and maximum fuzziness generalized operators were invented [21], and new parametric operator families were

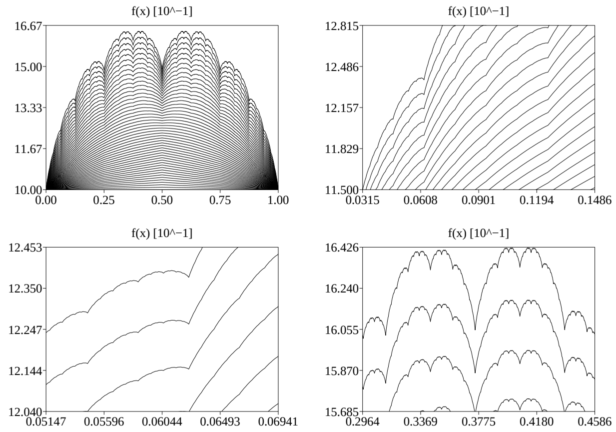

Fig. 1. The fractal-like graphs of the $f_\xi(x)$ functions for increasing $\xi \in \{0, 0.02, \ldots, 0.98\}$ values (increasing zooming ratios from left to right and from up to down).

introduced [22], etc.

To resolve the seemingly "antagonistic" contradiction between the successful practical applications and the theoretically proved "nowhere denseness properties" of *SC* methods one became apt to arrive at the conclusion that the problem roots in the fact that Kolmogorov's approximation theorem is valid for the very wide class of *continuous functions* that contains even very "*extreme*" elements at least from the point of view of the technical applications. (The first example of a function that everywhere is continuous but nowhere is differentiable was given by Weierstraß in 1872 [23].)

To highlight the "generality" of the concept of continuous functions another examples were given e.g. in [24] where a set of functions that are everywhere continuous in the open interval $(0,1)$, and almost everywhere are differentiable within this interval were presented. These functions are defined as a series built up from a set of "triangular functions" in which $\varphi_0(x) := 1$, $\varphi_1 := 2x$ if $x \in (0, 0.5)$, $1 - 2(x - 0.5)$ if $x \in [0.5, 1)$, etc. Each $\{\varphi_i | i \geq 1\}$ function takes the value 0 at $x = 0$, oscillates between 0 and 1 with constant derivatives of $\pm 2^i$, and cannot be differentiated at the halving ("break") points of the $(0,1)$ interval that can be expressed in the form of $x = \frac{m}{2^n}$, where m, n are positive integers. The functions under consideration are defined as $f_\xi(x) = \sum_{s=0}^{\infty} (\xi/2)^s \varphi_s(x)$ for the parameter $\xi \in [0, 1)$. The "graphs" of these functions are "visualized" in **Fig. 1** for increasing zooming ratios. The graphs evidently show fractal structure since in each zooming ratio they have some "zigzags". All this indicates that the problem of "dimensionality" may mean serious difficulties even in the case of one dimension *within the class of the continuous functions*.

Intuitively it can be expected that if we restrict our models to the far better behaving "everywhere differentiable" functions the problems with the dimensionality ab ovo could be evaded or at least reduced.

The first efforts in this direction were summarized in [25] in which the sizes of the necessary uniform structures used for developing *partial*, *temporal*, and *situation-dependent* models that needed continuous maintaining were definitely determined by the degree of freedom of the system to be controlled. These considerations were based on the modification of the *Renormalization Transformation*.

In the present paper, proceeding further on the path initiated in [26], this idea is systematically extended for *Single Input - Single Output (SISO)* by developing various *Parametric Fixed Point Transformations* more or less akin to the *Renormalization Transformation*. The applicability of the proposed method is illustrated by the use of a paradigm, namely by the control of the Ball-Beam System.

In the sequel at first the mathematical model of the paradigm is discussed, then the main idea of the novel adaptive control is developed in comparison with the traditional *Variable Structure / Sliding Mode Controller*, and finally simulation results and conclusions are delivered.

2. The Ball-Beam System

The accurate control of the "ball-beam system" in which a ball can roll on the surface of a beam the tilting angle of which is driven by some actuator is a physically interesting task. The motion of the ball essentially is determined by the tilting angle and the force of gravitation. This means that even if we are in the possession of a very strong actuator, the acceleration of the ball along the beam is limited by the above two factors. Since the directly controllable quantity is the torque determining the 2nd time-derivative of the angle tilting the beam, this system acts as a 4th order one in the sense that the 4th time-derivative of the ball's position along the beam is determined by the tilting torque. It has the following parameters: the momentum of the beam $\Theta_{Beam} = 2$ kg × m^2, the mass of the ball $m_{Ball} = 2$ kg, the radius of the ball $r = 0.05$ m, and the gravitational acceleration is $g = 9.81$ m/s^2. Via introducing the quantities $A = \Theta_{Beam}$, and $B = \Theta_{Ball}/r^2 + m_{Ball}$, the following equations of motion are obtained:

$$\begin{aligned} &A\ddot{\varphi} + m_{Ball} x \cos\varphi - m_{Ball} r \sin\varphi = Q \\ &B\ddot{x} + m_{Ball} g \sin\varphi = 0 \quad \ldots\ldots\ldots\ldots \quad (1) \end{aligned}$$

in which variable φ describes the rotation of the beam counter-clockwisely with respect to the horizontal position in rad units, and x in m denotes the distance of the ball from the center of the beam where it is supported. Variable Q in Nm describes the torque at the axis rotating the beam. This quantity consists of two different components: the torque directly exerted by the drive, and the contribution by the friction forces acting at the surface of the axle. In this paper this latter component is unknown by the controller, only the consequences of its existence in the trajectory tracking can be observed.

From Eq. (1) $\ddot{x}$ can be expressed as a function of φ. Since this angle cannot be made abruptly vary, following two derivations by time the 4th time-derivative of x can be expressed with $\ddot{\varphi}$ as follows:

$$x^{(4)} = \frac{m_{Ball}g}{B}\left(\sin\varphi\dot{\varphi}^2 - \cos\varphi\ddot{\varphi}\right). \quad . \quad . \quad . \quad . \quad . \quad (2)$$

In the possession of the desired $x^{(4)}$ value by the use of Eq. (2) and the 1st equation of the group (1) the necessary torque Q can be computed in principle. Normally it can be supposed that the parameters of the actual system are not precisely known. Instead of the actual parameters some model values are used as $\widetilde{A}$, and $\widetilde{B}$ constructed of the model values of the other parameters. On the basis of this rough model at first the *desired rotational acceleration of the beam* is estimated as

$$\ddot{\varphi}^{Des} = \frac{-\widetilde{B}s^{(4)}}{\widetilde{m_{Ball}g}\cos\varphi} + \tan\varphi\dot{\varphi}^2 - \Gamma_{\varphi}\frac{\partial\cosh^{2n}\left(\frac{\beta_{Pot}\varphi}{1.5}\right)}{\partial\varphi} - \Gamma_{\dot{\varphi}}\frac{\partial\cosh^{2n}\left(\frac{\beta_{Pot}\dot{\varphi}}{3}\right)}{\partial\dot{\varphi}} \quad . \, (3)$$

in which the last two terms limiting potentials are introduced for the *rotational angle* and for the *rotational velocity* of the beam. In the simulations $\beta_{Pot} = 5$ [*dimensionless*], and $n = 1$ [*dimensionless*] were used. The physical meanings of the potential terms are evident: the 1st term nonlinearly curbs the increase in $|\varphi|$ around 1.5 rad, the 2nd term curbs the angular velocity $|\dot{\varphi}|$ around 3 rad/s. Nearby the small $|\varphi|$ angles and small $|\dot{\varphi}|$ values the effects these "flat" potentials are negligible. This "moderated" value for $\ddot{\varphi}^{Des}$ is then substituted into the "model variant" of the 1st equation of the group (1) to calculate the necessary torque, Q^{Drive}. By properly setting the parameters Γ_{φ}, $\Gamma_{\dot{\varphi}}$, and β_{Pot} for the actual feedback policy these terms can guarantee that φ and $\dot{\varphi}$ remain within certain limits.

For describing the phenomenon of friction various models are available, e.g. [27–29]. The so called "static" models as the *Striebeck Model* establishes a simple approximate functional relationship between the relative velocity of the surfaces sliding on each other and the friction forces whenever this velocity v differs from zero as

$$F = -\mathrm{sign}(v)\left[F_C + F_S\exp\left(-\frac{|v|}{v_s}\right)\right], \quad v \neq 0. \quad (4)$$

This model does not give satisfactory description of the "sticking" phenomenon i.e. the observation that v stagnates at zero until the external force achieves an $F_C + F_S > F_C$ limit in its absolute value. It describes only the experienced behavior that the friction force decreases with increasing absolute value of the velocity. For numerical simulation of stiction this model has to be "completed" by the introduction of a fictitious small velocity region centered near zero. If this region is achieved the velocity is kept at exactly zero until the external force exceeds the limit $F_C + F_S$. This way of modeling is not very well substantiated on physical basis and the actual value of this velocity limit remains dubious. To provide some physical picture the so called *Tustin Model* can be introduced in which the variable "z" describes a kind of *internal deformation* of the connected surfaces, i.e. a hidden internal degree of freedom that has its own equation of state propagation as

$$\frac{dz}{dt} = v - \frac{\sigma_0|v|}{F_C + F_S\exp\left(-|v|/v_s\right)}z, \quad F = \sigma_0 z \quad . \quad (5)$$

therefore $\dot{z} = v\left[1 - \frac{F\times\mathrm{sign}(v)}{F_C + F_S\exp\left(-|v|/v_s\right)}\right]$. The simple picture behind this model is the supposition that some elastic deformation happens via small springs that partly are destroyed (disconnected) with higher displacements. Consequently z can be increased in its absolute value only to a velocity-dependent limit, and it stagnates at this value until the velocity changes its sign. This changing sign causes abrupt, discontinuous variation in $\dot{z}$, and a fast variation in z. In the LuGre model the above contribution is completed by a pure viscous term, and an additional one behind which the deformation of the bristles of some "brush" are hidden as physical models:

$$\frac{dz}{dt} = v - \frac{\sigma_0|v|}{F_C + F_S\exp\left(-|v|/v_s\right)}z$$

$$F = \sigma_0 z + \sigma_1\frac{dz}{dt} + F_v \times v \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

for which the proper direction of F has to be set in the applications, F_v describes the viscous friction coefficient, and σ_1 is a new parameter pertaining to the effect of the bending bristles. This model is physically complete in the sense that no any velocity limit of dubious interpretation must be introduced for its use. The behavior of the whole system is described by the dynamic coupling between the hidden internal and the observed degrees of freedom. Though the appropriate quantities in Eq. (6) were developed for linear motion and forces, it easily can be generalized for rotary motion in whin torques appear in the role of the forces, and rotational velocity is present instead of linear motion's velocity. The model given in Eq. (1) evidently can be completed via adding the additional torque of the friction to Q in it. In the sequel the *robust VS/SM control* and a *novel adaptive control* will be investigated *for compensating the effects of the model uncertainties and the friction directly unknown by the controllers.*

3. The Robust VS/SM and the Adaptive Controllers

In both cases it is plausible to introduce the linear operator $\Lambda := (d/dt + \lambda)$ with $\lambda > 0$, and apply its appropriate power to the trajectory tracking error. Really, if for an integer constant $k > 0$ and a function $f(t)$ the situation of $\Lambda^k f(t) = 0$ is achieved, the quantity $\Lambda^{k-1}f(t) \to 0$ exponentially as $\propto e^{-\lambda t}$. As soon as the situation of $\Lambda^{k-1}f(t) \approx 0$ has been approximated practically the quantity $\Lambda^{k-2}f(t)$ starts to decay exponentially as $\propto e^{-\lambda t}$, etc. Finally the situation of $f(t) \to 0$ is achieved. It is worth noting that for this convergence it is not necessary to guarantee the exact value of λ. Any positive $\tilde{\lambda} \approx \lambda$ works well with a little bit different speed of convergence. In the case of an m^{th} order system when $d^m x/dt^m$ can directly be manipulated by the drive(s) the controller may have different *"ambitions."* The typical control solutions can be classified according to their *"ambitions."*

3.1. The Traditional Robust VS/SM Controller

Whenever only a very rough system model is available $d^m x/dt^m$ cannot exactly be prescribed. Instead of that by the use of Λ *and the trajectory tracking error h an "error metrics"* can be defined as $S(t) := \Lambda^{m-1} h(t)$, and some attempt is done to drive S to zero *during finite time*. Since dS/dt contains $d^m x/dt^m$ a plausible choice is to approximate the *"desired"* situation

$$\dot{S}^{Des} \approx -K \operatorname{sign}(S) \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (7)$$

with satisfactorily big positive constant K. Though a precisely prescribed K value cannot be achieved in the lack of the exact system model, the situation $0 < K_{lim} \leq K$ can be achieved on the basis of a rough system model. Normally only some rough over-estimation of the necessary driver action is obtained, therefore, as a consequence, the sign of the achieved S fluctuates that also makes the driving force/torque fluctuate. This fluctuation is the phenomenon of *"chattering"* so typical in the case of the *VS/SM Control*. A possibility for reducing chattering is smoothing the variation of $\dot{S}$ in Eq. (7) as

$$\dot{S}^{Des} = -K \tanh\left(\frac{2S}{w}\right) \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (8)$$

in which w is a properly chosen *"width parameter"* within which the jump in Eq. (7) is smoothed. This smoothing evidently decreases the speed of the decay of S, therefore *degrades the accuracy of trajectory tracking*. A possibility for improving this situation is the application of fractional order differential equation for the desired decay of S.

3.2. The Fractional Order Robust VS/SM Controller

Though the formal mathematical idea of introducing non-integer order derivatives goes back to the end of the 17th century in a communication between L'Hospital and Leibniz in 1695 [30], significant development of the concept was achieved in the 19th century [31–33]. Due to the lack of its physical interpretation the first applications in Physics appeared only in the 20th century, in connection with visco-elastic phenomena [34, 35]. The topic later obtained quite general interests [36–38], and also obtained new applications in material science [39], analysis of earth-quake signals [40], control of robots [41], description of diffusion [42], etc.

The concept of fractional derivatives has many, only more or less equivalent definitions, e.g. by Riemann-Liouville, Caputo, Grüwald-Letnikov, Hadamard, Marchaud, Riesz, etc. In this paper we use the form invented by Caputo for its lucidity and simplicity:

$$\left({}_{a}^{C}D_{t}^{\beta} u\right)(t) := \frac{1}{\Gamma(1-\beta)} \int_{a}^{t} \left[\frac{du}{d\tau}\right] (t-\tau)^{-\beta} d\tau \qquad (9)$$

in which $\beta \in (0,1)$ means the *order of differentiation*, and $t \geq a$. This definition re-integrates the 1st integer derivative with a kernel function that is singular at the upper bound of the integration. This singularity enhances the contribution of the "present time", while the "tail" of the kernel works as a filter yielding some effects of the past, too. In the practical realization of Eq. (9) the lower limit of the integration is $t-T$ that corresponds to a finite "memory of length" T. Furthermore, for the numerical approximation of the integral with singular integrand the following formula can be used: the full interval of the integration of length T is divided into small sub-intervals of length δ, during which the reintegrated derivative is supposed to be approximately constant:

$$\begin{aligned} {}_{t-T}^{\;\;C}D_{t}^{\beta} u \equiv \frac{d^{\beta} u}{dt^{\beta}} &\approx \frac{\delta^{-\beta+1}}{\Gamma(2-\beta)} \dot{u}(t) + \frac{\delta^{-\beta+1}}{\Gamma(2-\beta)} \\ &\quad \times \sum_{s=1}^{T/\delta} \left[(s+1)^{-\beta+1} - s^{-\beta+1}\right] \dot{u}(t-s\delta) \\ &= \sum_{s=0}^{T} G_s \dot{u}(t-\delta s). \end{aligned}$$

This numerical approximation has the trivial properties that for *constant* $\dot{u}$ it yields *constant fractional derivative*, for $\dot{u} \equiv 0$ it yields *zero*, and for $\beta = 1$ it just yields the 1st *integer order derivative*.

For a fixed memory-length T a fractional integro-differential equation can be defined as ${}_{t-T}^{\;\;C}D_{t}^{\beta} u(t) = f(t)$ in which instead of the concept of the *initial value* that of the *preceding history*, that is the $u(t)$ values in the $[t-T, t]$ interval conveys the necessary information for calculation, and $f(t)$ serves a kind of *external excitation*. This equation describes the behavior of a system of finite memory. For *discrete fixed time-resolution* this integro-differential equation can be substituted by an equation of differences as

$$\left({}_{t-T}^{\;\;C}D_{t}^{\beta} u\right)_{k+1} \approx \sum_{s=0}^{T} G_s \frac{u_{k+1-s} - u_{k-s}}{\delta} = f_{k+1}. \qquad (10)$$

For instance for a constant excitation $f \equiv K$ Eq. (10) can be rearranged as

$$\dot{u}_{k+1} := \frac{u_{k+1} - u_k}{\delta} = \frac{K}{G_0} - \sum_{s=1}^{T} \frac{G_s}{G_0} \dot{u}_{k+1-s} \quad . \; . \; (11)$$

defining a *Cauchy Sequence* since for the "*velocity steps*" $w_{k+1} := \dot{u}_{k+1} - \dot{u}_k$

$$|w_{k+1}| = \left| \sum_{s=1}^{T} \frac{G_s}{G_0} w_{k+1-s} \right| \leq M \max_{s=1}^{T} |w_{k+1-s}| \quad . \; (12)$$

in which $0 < M = |G_1/G_0| < 1$. This means that the absolute value of the velocity step in the $(k+1)^{\text{th}}$ time step certainly is smaller than the maximum of its counterparts in

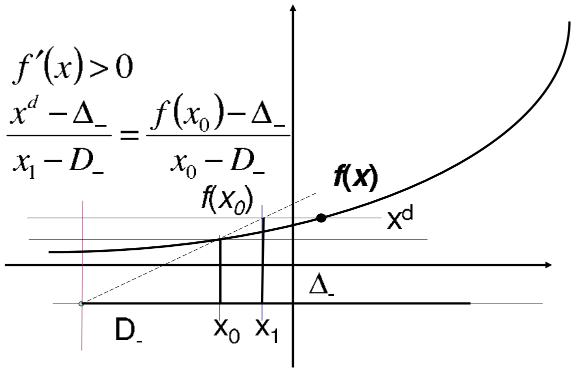

Fig. 2. Geometric illustration of the proposed fixed-point transformation generated by function $h(x|x^d,D_-,\Delta_-)$.

the $\{k,k-1,\ldots,k-T\}$ interval. Via stepping ahead $T+1$ steps the disjoint $\{k+T+1,\ldots,k+1\}$ interval is obtained within which the maximal absolute step is certainly smaller than or equal to M times of the maximum of the previous interval. Since $0<M<1$ $w_z \to 0$ as $z\to\infty$. Due to the completeness of the $\Re$ space with the norm $|\cdot|$ this means that the sequence of the velocities is convergent to a finite value $\dot{u}_k \to v$. Via substituting that into Eq. (11) the limit value can be obtained as $\dot{u}=v=\frac{K}{G_0+\sum_1^T G_s}$. This convergence property gives definite mathematical meaning to modifying the required damping of the error metrics as

$$ {}_{t-T}^{\;C}D_t^{\beta} S^{Des} = -K\tanh\left(\frac{2S}{w}\right) \quad \ldots\ldots\ldots (13)$$

in which the $\{G_s\}$ coefficients for $\{s=0,\ldots,T\}$ serve as frequency filters "further conveying" the past values in a damped manner.

3.3. Iterative Adaptive Control with Fixed Point Transformation

Each control task can be formulated by using the concepts of the appropriate *"excitation" Q* of the controlled system to which it is expected to respond by some prescribed or *"desired response" r^d*. The appropriate excitation can be computed by the use of some *inverse dynamic model $Q=\varphi(r^d)$*. Since normally this inverse model is neither complete nor exact, the actual response determined by the system's dynamics, ψ, results in a *realized response r^r* that differs from the desired one: $r^r=\psi(\varphi(r^d))\equiv f(r^d)\neq r^d$. Now restrict ourselves to *SISO* systems, and for obtaining proper deformation of the input value for obtaining better response, consider the following functions that are suggested by geometrically similar triangles as given e.g. in **Fig. 2**:

$$\begin{aligned} h(x|x^d,D_-,\Delta_-) &= \frac{x^d-\Delta_-}{f(x)-\Delta_-}(x-D_-)+D_- \\ g(x|x^d,D_-,\Delta_+) &= \frac{f(x)-\Delta_+}{x^d-\Delta_+}(x-D_-)+D_-. \quad (14) \end{aligned}$$

For these functions it evidently is true that if $f(x_\star)=x^d$ then $g(x_\star|x^d,D_-,\Delta_+)=x_\star$ and $h(x_\star|x^d,D_-,\Delta_-)=x_\star$, that is *the solution of the control problem just is the fixed point of these functions*. In this manner the original control problem, i.e. finding a properly deformed input $x_\star$ is transformed to finding the solution of a fixed point problem. Fixed point problems in general have the advantageous feature that they can be solved via simple iteration provided that this iteration is convergent as $x_{n+1}=g(x_n|x^d,D_-,\Delta_+)$ or $x_{n+1}=h(x_n|x^d,D_-,\Delta_-)$. As is well known for the convergence of such iteration *contractivity* of functions h and g within a region (i.e. within an open connected set) is satisfactory condition. Really, consider the sequence of points $\{x_0,x_1=\Psi(x_0),\ldots,x_{n+1}=\Psi(x_n),\ldots\}$ obtained via iteration! Let us suppose that this series converges to some limit value: $x_n\to x_\star$. In order to apply iterations let us consider the set of the real numbers as a *linear normed space with the common addition and multiplication with real numbers, and with the absolute value $|\bullet|$ as a norm*! It is well known that this space is *complete*, i.e. it is a *Banach Space* in which the *Cauchy Sequence* are convergent. Due to that, using the norm inequality it is obtained that

$$\begin{aligned} |\Psi(x_\star)-x_\star| &\le |\Psi(x_\star)-x_n|+|x_n-x_\star| \\ &= |\Psi(x_\star)-\Psi(x_{n-1})|+|x_n-x_\star|. \quad . \quad (15) \end{aligned}$$

It is evident from Eq. (15) that if Ψ is continuous then $\Psi(x_\star)=x_\star$, that is the desired fixed point is found by the iteration because the right hand side of Eq. (15) converges to 0 as $x_n\to x_\star$. The next question is giving the necessary or at least a *satisfactory condition of this convergence*. It also is evident that for this purpose contractivity of $\Psi(\bullet)$, i.e. the property that $|\Psi(a)-\Psi(b)|\le K|a-b|$ with $0\le K<1$ is satisfactory since it leads to a *Cauchy Sequence* ($|x_{n+L}-x_n|\to 0$ as $n\to\infty$ $\forall L\in\mathbb{N}$):

$$\begin{aligned} |x_{n+L}-x_n| &= |\Psi(x_{n+L-1})-\Psi(x_{n-1})|\le\cdots \\ &\le K^n|x_L-x_0|\to 0. \quad \ldots\ldots\ldots (16) \end{aligned}$$

For a differentiable function $\Psi(\bullet)$ of derivative $|\Psi'|\le K<1$ the satisfactory condition of contractivity evidently holds:

$$\begin{aligned} |\Psi(a)-\Psi(b)| &= \left|\int_a^b \Psi'(x)dx\right| \\ &\le \int_a^b |\Psi'(x)|dx \le K|a-b| \quad . \quad (17) \end{aligned}$$

that means that if Ψ is flat enough around the fixed point the iteration will converge to it. It is easy to show that the derivatives of the functions g and h in Eq. (14) in the fixed point $x_\star$ take the following form:

$$\begin{aligned} h'(x_\star) &= 1-f'(x_\star)\frac{x_\star-D_-}{x^d-\Delta_-} \\ g'(x_\star) &= 1+f'(x_\star)\frac{x_\star-D_-}{x^d-\Delta_+}. \quad \ldots\ldots\ldots (18) \end{aligned}$$

Supposing that the system to be controlled is not singular, by manipulating the model function it is easy to guarantee that $|f'(x_\star)|$ is small enough in a whole region around $x_\star$. According to Eq. (18) this means that if $f'(x)>0, x>D_-$, $x^d<\Delta_+$, and $x^d>\Delta_-$ then the conditions $-1<g'<1$

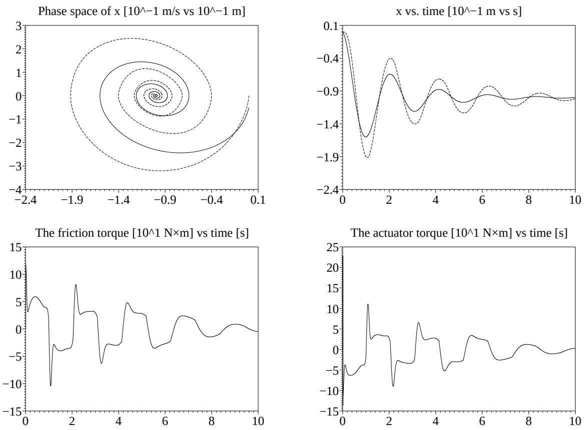

Fig. 3. The operation of the *"common controller"*: phase space of the translation of the ball along the beam (upper left), the nominal and the computed trajectories of the ball (upper right), the friction torque (lower left), and the driving torque (lower right) vs. time.

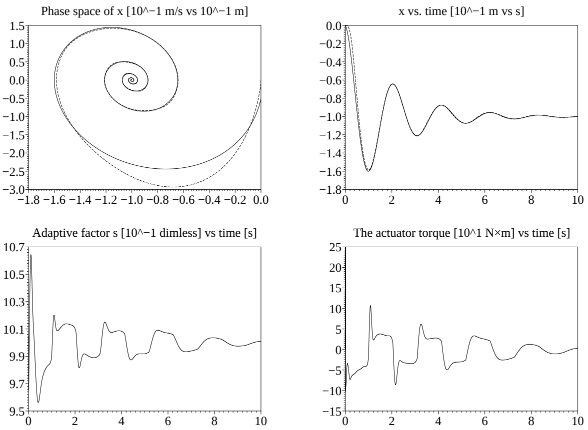

Fig. 4. The operation of the *"adaptive controller"*: phase space of the translation of the ball along the beam (upper left), the nominal and the computed trajectories of the ball (upper right), the adaptive parameter (lower left), and the driving torque (lower right) vs. time.

and $-1 < h' < 1$ can be met. It is worth noting these functions alone or their combination can be applied for control purposes, too. In the here presented simulations the use of these two functions are combined.

It is interesting, too, that for very big $|\Delta_\pm|$ $g \approx h \approx x$, therefore the control parameters can easily be set experimentally by starting with a "flat" inverse dynamic model, with big $\Delta_+ = -\Delta_-$ values and a near-zero D_- value. In the sequel the operation of these controllers are investigated via simulations.

4. Simulation Results

For modeling the effect of friction very drastic parameter settings was chosen as $\sigma_0 = 6000$ Nm/rad, $\sigma_1 = 2000$ Nms/rad, $F_C = 10$ Nm, $F_S = 20$ Nm, $F_v = 200$ Nms/rad. Regarding the angular acceleration of the tilting angle of the beam a common "Computed Torque Controller's" operation is described in **Fig. 3** that uses the imprecise model also neglecting the friction. The diagrams make it clear that the driving torque is mainly used for the compensation of the drastic friction effects. The other dynamic properties of the system practically are not taken into account by the controller. The damping in the amplitude of the nominal motion was intentionally chosen to show that the effects of the friction seriously and nonlinearly depend on the velocity and the amplitude of the nominal motion.

The *adaptive counterpart* of the control for the same nominal motion and friction parameters are depicted in **Fig. 4** for the near-optimal control parameter settings of $\Delta_- = -8 \times 10^4$, $\Delta_+ = 8 \times 10^4$, $D_- = -10^3$ m/s^4.

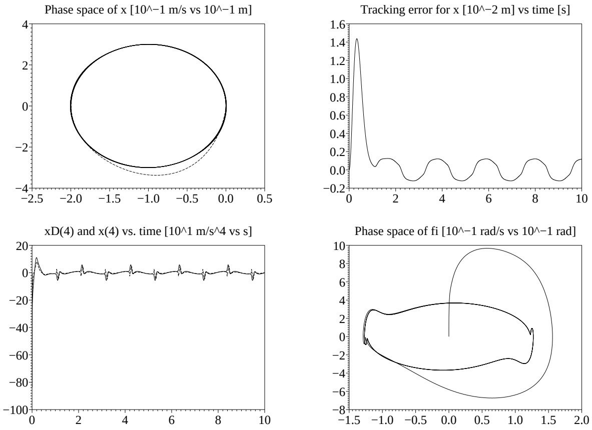

Fig. 5. The operation of the adaptive controller at near-optimal parameter settings: phase space of the translation of the ball along the beam (upper left), the tracking error (upper right), the desired $(d^4x/dt^4)^d$ and the attained d^4x/dt^4 (lower left) vs. time, and the phase space of the tilting angle (lower right).

It has to be noted that during one control cycle only one step of iteration was executed that is the control signals $x^{(4)}(t_n) = g(x^{(4)}(t_{n-1})|x^{(4)^d}(t_n), D_-, \Delta_+)$ or $x^{(4)}(t_n) = h(x^{(4)}(t_{n-1})|x^{(4)^d}(t_n), D_-, \Delta_-)$ were applied depending on the sign of the desired action. If the adaptation is faster than the dynamics of the system to be controlled appropriate result can be expected. This approach is similar to the application of *Cellular Neural Networks (CNN)* for image processing. In this case the concept of *"Complete Stability"* means that a static input picture is mapped to a static output picture following a short dynamic transition of the physical state of the CNN. If the input picture is not static but varies "slowly" in comparison with the "speed" of the CNN's internal dynamics varying picture is mapped to varying output [43]. In spite of using a single step during one control cycle the improvement in the trajectory tracking and phase trajectory tracking achieved by the use of the adaptive control is quite illustrative. (In the figure the *"adaptive parameter"* is defined as $s(t_n) =$

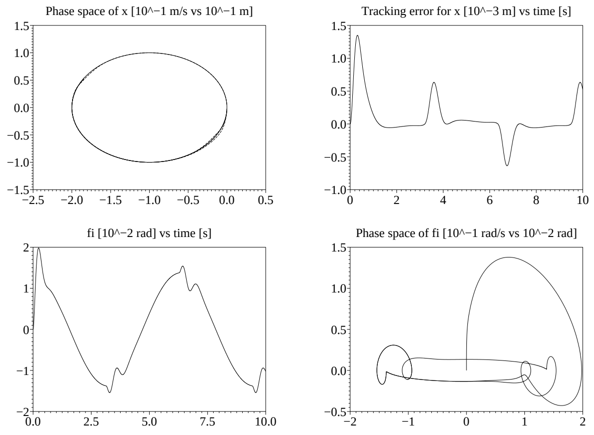

Fig. 6. The operation of the adaptive controller at near-optimal parameter settings for slow nominal motion: phase space of the translation of the ball along the beam (upper left), the tracking error (upper right), the variation of the tilting angle φ (lower left) vs. time, and the phase space of the tilting angle (lower right).

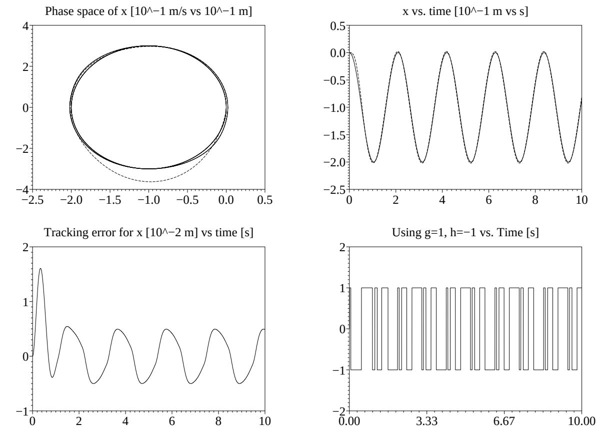

Fig. 7. The operation of the adaptive controller at suboptimal parameter settings: phase space of the translation of the ball along the beam (upper left), the nominal and the computed trajectories of the ball (upper right), the tracking error (lower left) vs. time, and the adaptive transformation chosen vs. time (lower right).

$\prod_{u=1}^{n} \frac{x^{(4)d}(t_u)-\Delta_-}{f(x^{(4)}(t_{u-1}))-\Delta_-}$ or $s(t_n) = \prod_{u=1}^{n} \frac{f(x^{(4)}(t_{u-1}))-\Delta_+}{x^{(4)d}(t_u)-\Delta_+}$, or the cumulative mixture of these factors.) According to **Fig. 4** the adaptive factor well mirrors and compensates the effects of friction.

To further investigate the adaptive controller tracking a non-damped nominal trajectory with the above given "near optimal" parameter settings is depicted in **Fig. 5**. It can well be seen that the desired and the attained 4th derivatives of x are really very close to each other, and that the trajectory tracking error is under 2 mm. This observation is in harmony with the expectation that the fixed point has a basin of attraction therefore it "pulls" the solution with itself as $x^{(4)d}$ varies in time.

The effect of the strongly non-linear part of the friction is more apparent in the case of a relatively slow motion (**Fig. 6**).

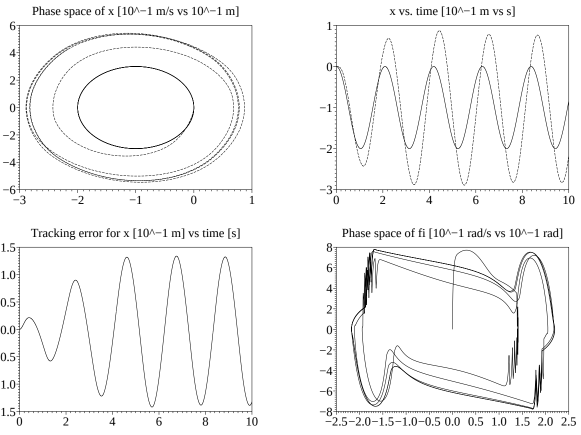

Fig. 8. The operation of the integer order VS/SM controller based on the rough system model neglecting friction effects without applying anti-chattering: phase space of the translation of the ball along the beam (upper left), the nominal and the computed trajectories of the ball (upper right), the tracking error (lower left), and the phase space of the rotation of the beam (lower right) vs. time.

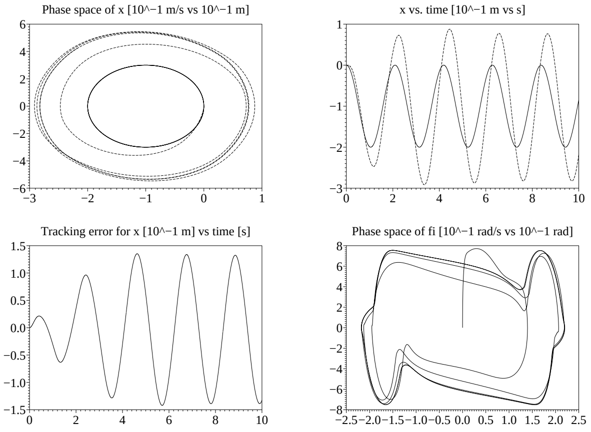

Fig. 9. The operation of the integer order VS/SM controller based on the rough system model neglecting friction effects and applying anti-chattering width parameter $w = 3$ m/s^3: phase space of the translation of the ball along the beam (upper left), the nominal and the computed trajectories of the ball (upper right), the tracking error (lower left), and the phase space of the rotation of the beam (lower right) vs. time.

To illustrate the robustness of the method the sub-optimal counterpart of **Fig. 5** [$D_- = -500$, $\Delta_- = -1.6\times\ 10^5$, $\Delta_+ = 1.6\times 10^5$] can be seen in **Fig. 7**. It is well illustrated that in spite of the drastic variation of the control parameters acceptable solution was obtained.

In **Fig. 8** the results for integer 1st order derivatives without using anti-chattering are presented. It is evident from the phase space of the tilting angle of the beam that some chattering is initiated in the control, and in spite of that the tracking accuracy is very bad.

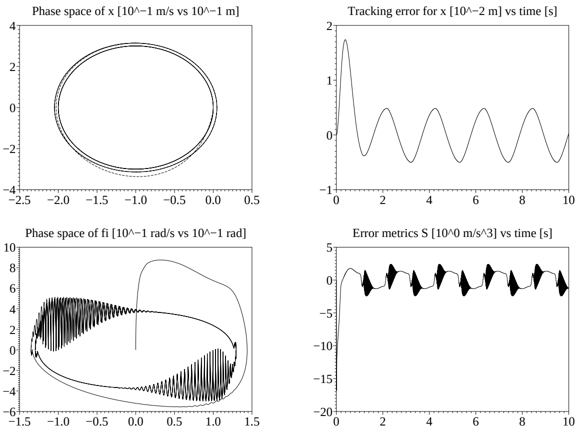

Fig. 10. The operation of the $\beta = 0.9$ order VS/SM controller based on the rough system model neglecting friction effects and applying anti-chattering width parameter $w = 3$ m/s^3: phase space of the translation of the ball along the beam (upper left), the tracking error (upper right), the phase space of the tilting angle of the beam (lower left), and the error metrics vs. time (lower right).

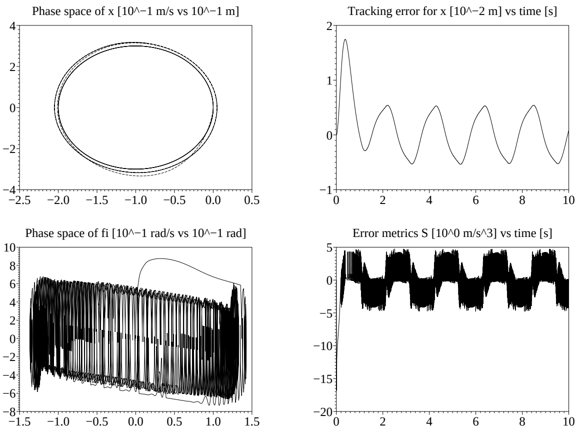

Fig. 12. The operation of the $\beta = 0.9$ order VS/SM controller allowing chattering (counterpart of **Fig. 10**): phase space of the translation of the ball along the beam (upper left), the tracking error (upper right), the phase space of the tilting angle of the beam (lower left), and the error metrics vs. time (lower right).

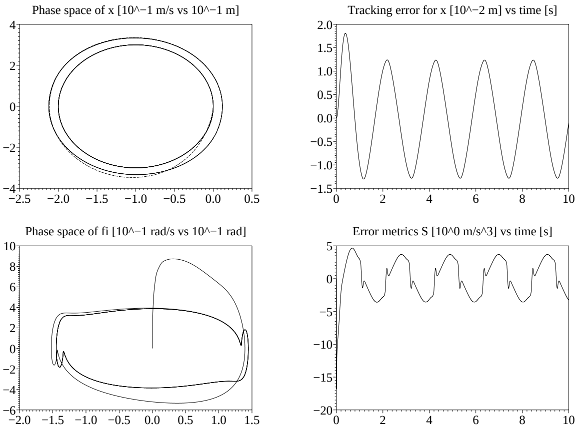

Fig. 11. The operation of the $\beta = 0.9$ order VS/SM controller based on the rough system model neglecting friction effects and applying increased anti-chattering width parameter $w = 8$ m/s^3: phase space of the translation of the ball along the beam (upper left), the tracking error (upper right), the phase space of the tilting angle of the beam (lower left), and the error metrics vs. time (lower right).

To evade this chattering in **Fig. 9** the results for integer 1st order derivatives with anti-chattering width parameter of $w = 3$ m/s^3 are presented. It is evident that no chattering occurred in this case, and the tracking accuracy has not been considerably degraded. However, it remained very bad. The effect of friction can well be realized in the phase space of the rotation of the beam.

The $\beta = 0.9$, $\delta = 10^{-3}$ s, $T = 10$ ms fractional order counterpart of the motion depicted in **Fig. 9** is given in **Fig. 10**. The improvement in the tracking accuracy is illustrative, however it cannot reach the quality of the adaptive controller. Furthermore, in certain parts of the phase trajectory appearance of some chattering can be observed. To evade chattering the width parameter has been increased to $w = 8$ m/s^3 in **Fig. 11**. It is worth noting that the tracking accuracy has been degraded to a considerable extent. Furthermore, variation of the error metrics S in this figure well mirrors the effects of the friction.

For further improvement in the tracking accuracy a possibility within the frames of the VS/SM Control is allowing the occurrence of chattering i.e. using Eq. (7) instead of Eq. (8). The appropriate counterpart of the control depicted by **Fig. 10** is given in **Fig. 12**.

It can well be seen that the improvement in the tracking accuracy is not negligible, though it is achieved at the costs of having strong chattering in the tilting angle of the beam. Furthermore, the tracking accuracy does not reach that of the near-optimal adaptive control that works without chattering.

5. Conclusions

In this paper the operations of the traditional VS/SM controller, the fractional order VS/SM controller, and the fixed point transformations based adaptive controllers were compared to each other in the control of a "Ball-Beam System" serving as a typical nonlinear paradigm.

The fractional VS/SM control utilizes the memory and internal transient dynamics manifesting itself in fractional order differential equations. The adaptive approach was elaborated to be some "alternative counterpart" of the modern Soft Computing approaches representing universal approximators in the set of continuous functions. Instead of the set of continuous functions that may contain very "extremely behaving" elements it aims at the use of differentiable functions.

It was found that the use of fractional order derivatives considerably improves the accuracy of the integer order VS/SM controllers. However, good accuracy can be obtained only if chattering is allowed in the control.

The iterative adaptive controller was found to be quite efficient without causing any chattering. Furthermore, it can be also stated that it is even simpler than the robust fractional order VS/SM controllers. Though the paradigm considered essentially behaved as a Single Input Single Output (SISO) system, it is expected that by the use of the concepts of "obtuse" and "acute" angles these considerations can be extended to Multiple Input Multiple Output (MIMO) systems. For this further research seems to be necessary.

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