Paper:

Graph/Knot Theoretical Analysis and Generation for Impossible Figures

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An algorithm to represent impossible multibar figures and their subclass of torus figures is proposed based on graph and knot theory. A multibar type graph, which is an abstract concept of multibar figures, is defined by the junction graph that represents the connections of the lines. It is shown that the junction graph is able to characterize multibar figures where this characterization is realized according to the type of the multibar type graph. An automatic drawing system of torus figures is also presented by analyzing junction graphs that construct the shapes of corners of torus figures. The proposed method aims a basic tool for experiments in visual psychology and possible/impossible figures generation.

Keywords: graph theory, knot theory, impossible figure, computer vision, visual psychology

1. Introduction

Studies of impossible figures that are evident examples of illusionism are mainly investigated by psychologists based on the mechanisms of illusion. A mathematical method is, however, proposed by Cowan [2, 3] based on braid theory to express and classify line drawing of impossible figures. The target is a subclass of the most important class of impossible figures in visual psychology, which are obtained by twisting figures connecting quadrangular prisms with the uniform rectangular cross-sections in a toroidal form. It is, however, still an interesting problem to determine what kind of geometrical properties such a classification has. While shape from line drawings as an issue of their interpretation is important research for application in 3D modeling and pattern recognition [9–12]. There are algebraic and combinatorial approaches to determine whether a line drawing figure, all of which vertices are consistent with the dictionary of trihedral vertices [5, 7], is an impossible figure based on a method of the shape from line drawings [6, 8]. These methods are applied to line drawings already pictured, not to automatic generation.

This paper proposes a well-defined representation of impossible multibar type figures and an algorithm for an automatic generating system of arbitrary torus figures based on graph and knot theory. A real (or possible) multibar figure is defined as a real object obtained by connecting the ends of quadrangular prisms. Impossible figures obtained by twisting and crossing them are said to be impossible (or unreal) multibar figures. A new mathematical representation for both possible and impossible multibar figures, called multibar figures, is proposed using graph theory. A multibar type graph is defined as an abstract concept of a multibar figure. It is the combined notion of a bar graph corresponding to lines of multibar figure and a junction graph corresponding to connections of their ends. An algorithm of the automatic generating system is constructed for arbitrary torus figures that comprise a subset of multibar figures and a mainly investigated object in the area of impossible figures. The input is a torus type graph that is an element of a subset of multibar type graphs, and the outputs are corresponding line drawings. A classification of torus figures is also introduced using knot theory. Because spatial embeddings [4] of a torus type graphs are links [1], corresponding figures are classified by the number of components and the number of edges of every component.

Constraints and topological properties of those figures are shown by analyzing generated line drawings and the generation process. Constraints of graphs corresponding to line drawings are shown by using constraints of junction graphs. Euler numbers [4] are calculated and relations between torus figures and link structures are shown by analyzing embeddable surfaces of a graph.

The proposed method aims a basic tool for experiments in visual psychology and possible/impossible figures generation that is expandable to nonresearch fields, including art and education.

Multibar type graphs, torus type graphs and link structures are defined in Section 2. A method for generationing torus figures from torus type graphs and for clarifying relations between generated line drawings and link structures is proposed in Section 3. Examples of generated

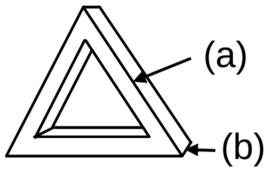

Fig. 1. Example of a possible multibar figure. (a) Longitudinal edge (b) Transversal edge.

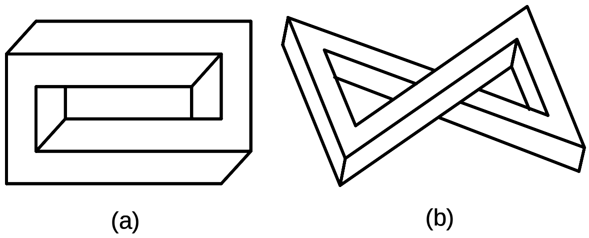

Fig. 2. Impossible multibar figures resulting from (a) twisting and (b) crossing.

line drawings and their scale-free invariants are given in Section 4.

2. Multibar Type Graph and Link Representation

2.1. Multibar Figure

A possible multibar figure is defined as a projection – parallel projection, general position [6]– of a polyhedron generated by connecting quadrangular prisms, which are prisms with the same rectangular cross-sections (**Fig. 1**). A line that corresponds to a generating line of a quadrangular prism is called a longitudinal edge (**Fig. 1**), while a line across two longitudinal edges at a connected part of two prisms is called a transversal edge (**Fig. 1**).

Figures that look like to be produced by twisting or crossing original multibars are called impossible multibar figures (**Fig. 2**). The components of real multibar figures such as longitudinal edges and transversal edges are also used in impossible multibar figures.

Mathematically speaking, a statement that "the figure looks like real multibar figure." is too intuitive. Therefore the following two hypotheses are formulated to cast the discussion in a more formal framework:

(Hypothesis 1) Every longitudinal edge including hidden edges is a projection of a generating line of quadrangular prisms with the uniform rectangular cross-sections.

(Hypothesis 2) Every vertex is compatible with the vertex dictionary [5, 7] of trihedral vertices.

Hypothesis 1 is a constraint for a figure to look like a connected figure of quadrangular prisms and Hypothesis 2 is a constraint for a figure to look like a polyhedron.

Both possible and impossible multibar figures are referred to as multibar figures.

2.2. Multibar Type Graph

Bars are defined as an abstract concept of quadrangular prisms and junctions as connections of bars. A multibar type graph based on these is an abstract reflection of multibar figures. To introduce several basic definitions:

Definition 1: Bar

Let X be a denumerable set, and V be a set of 8 different elements of X. Suppose a subset BS of 4 different elements of V, and a bijection $f : BS \to V - BS$ has been defined. The triple

$$B = (V, BS, f) \quad \ldots\ldots\ldots\ldots\ldots \quad (1)$$

is a bar, and $\mathbf{BS} = \{BS, f(BS)\}$ is bases of B.

Let B be a bar, a graph on vertices V coming with edges $BE = \{\{v, f(v)\} | v \in BS\}$, that is, a graph $BG = (V, BE)$ is a bar graph. If a finite set of bars

$$\mathbf{B} = \{B_n = (V_n, BS_n, f_n) | n \in \{1, 2, \cdots N\} = \mathbf{N}\} \quad (2)$$

satisfies the condition $^{\forall}n, n' \in \mathbf{N}\ (n \neq n')\ V_n \cap V_{n'} = \phi$, then $\mathbf{B}$ is a bar set or an N-bar (where $|\mathbf{B}| = N$).

Definition 2: Junction

Let $\mathbf{B}$ be an N-bar ($N \geq 2$), and let $\mathbf{S}$ ($\subset \mathbf{B}$) be a K-bar ($K \geq 2$). The union of the odd base of each bars $JV = \bigcup_{k \in K} BS_k$ ($BS_k \in \mathbf{BS}_k$) is given.

If a set JE ($\subset \{\{v, v'\} \subset JV | v \neq v'\}$) of two different elements of JV satisfies the following conditions

(J1) Let $F : JV \to 2^{JE}$ be a $F(v) = \{e \in JE | v \in e\}$

$$^{\forall}v \in JV \quad 1 \leq |F(v)| \leq 2$$

(J2) $^{\forall}e = \{v, v'\} \in JE \quad v \in BS_k \Rightarrow v' \notin BS_k$

(J3) $^{\forall}e_1 = \{v_1, v_1'\}, e_2 = \{v_2, v_2'\} \in JE \ {}^{\forall}v_1, v_2 \in BS_k \quad v_1 \neq v_2 \Rightarrow v_1' \neq v_2'$

(J4) $^{\forall}JE' \subset JE\ (JE' \neq JE)\ JE'$ does not satisfy (J1)-(J3).

a graph formed on vertices JV and having edges JE, that is,

$$JG = (JV, JE) \quad \ldots\ldots\ldots\ldots\ldots \quad (3)$$

is a junction or K-junction (where $K = |\mathbf{S}|$) on $\mathbf{B}$. If $K = 2$, the junction is called a simple junction.

$B_k \in \mathbf{S}$ has the K-junction JG at the base BS_k, and $B_k \in \mathbf{S}$ and $B_{k'} \in \mathbf{S}$ ($k' \neq k$) are adjacent.

A set of M different junction on $\mathbf{B}$, that is,

$$\mathbf{JG} = \{JG_m | JG_m : \text{junction},\ m \in \{1, 2, \cdots M\} = \mathbf{M}\} \quad (4)$$

is junction set or M-junction set (where $M = |\mathbf{JG}|$) on $\mathbf{B}$.

Definition 3: Multibar type graph

Let $\mathbf{B}$ be an N-bar ($N \geq 2$). If an M-junction set $\mathbf{JG} = \{JG_m | JG_m = (JV_m, JE_m)\}$ on $\mathbf{B}$ satisfies

(M) $^{\forall}BS \in \bigcup_{n \in \mathbf{N}} \mathbf{BS}_n \ {}^{\exists 1}m \in \mathbf{M} \quad BS \subset JV_m$

then, a graph on vertices $^{\forall}MV \in \bigcup_{n \in \mathbf{N}} V_n$ having edges $^{\forall}ME \in \bigcup_{n \in \mathbf{N}} BE_n \cup \bigcup_{m \in \mathbf{M}} JE_m$, that is,

$$G_M(\mathbf{JG}) = (MV, ME) \quad \ldots\ldots\ldots\ldots \quad (5)$$