Paper:

A Family of Polymodal Systems and its Application to Generalized Possibility Measures and Multi-Rough Sets

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Polymodal systems generally have large areas of applications to theoretical computer science including the theory of programming, while other applications are not yet fully explored. In this paper we consider a family of polymodal systems with the structure of lattices on the polymodal indices. After investigating theory of the polymodal systems such as the completeness, we study two applications. One is generalized possibility measures in which lattice-valued measures are proposed and relations with the ordinary possibility and necessity measures are uncovered. Second application is consideration of an information system as a table such as the one in the relational database. It is known that rough sets are used to discover regularities from such information tables. Applying polymodal logic concept, we generalize rough sets which are called multi-rough sets here. Our consideration is mainly to establish theoretical frameworks in these two application areas and hence no real examples are shown here.

Keywords: polymodal system, generalized possibility measure, multi-rough set

1. Introduction

Polymodal logic which is also called multimodal logic handles more than one sort of modality in the modal logic. It has been known that polymodal systems [8] have wide areas of applications due to its strong descriptive capability. A main area of applications is in theoretical computer science where many studies have been done on theory of programming and communications. However, applications to system science are still few and there are many possibilities of research opportunities.

Fuzzy sets [10] and rough sets [6, 7] are now main tools in both computational intelligence and system science. It is natural to consider applications of polymodal logic to these two fields. We thus study a family of polymodal systems in which indices to describe polymodality form lattices, and study its theory such as the completeness.

We show two applications. One is a generalization of the possibility theory [2, 11], where we propose lattice-valued possibility and necessity measures, and investigate theoretical relations with the ordinary possibility and necessity measures.

Second application is a generalization of rough sets [6, 7]. Rough sets are used to discover regularities in an information table. It is also known that rough sets are closely related to the modal system S5 [1] and hence theoretical results in modal logic are applicable to rough sets. As a polymodal system generalizes the basic monomodal logic, a parallel generalization of rough sets should be considered, which we propose and call multi-rough sets whose name is after multimodal logic.

Notice also that an information table has been studied in relational database. Hence notations in relational database can be used in multi-rough sets. We thus study typical multi-rough sets using the framework of relational database and use terms therein.

Throughout this paper, our main concern is to develop a theory and hence no real examples are given.

It will be shown that these two applications can be put into the same framework of indexed rough approximations, and therefore can both be called multi-rough sets.

Proofs of all propositions are summarized in the appendix, as some readers may not be interested in that technical part.

2. Indexed Rough Approximations

Let W be a set of objects and Λ be a lattice [5] to describe structures on W. Hence W and Λ are two different types of sets. The lattice has the ordering $\succeq$ and the two operations of $\sup(\alpha,\beta)$ and $\inf(\alpha,\beta)$.

A family of binary relations, $R(\alpha)$, $\alpha \in \Lambda$, is considered. Each relation is used to define rough approximations. $R(\alpha)$ is assumed to be reflexive, but may or may not be an equivalence relation in general. When $R(\alpha)$ is an equivalence relation, the standard rough approximations are obtained [7], while if the relation is not necessarily an equivalence, we have generalized rough approximations.

We assume the following.

Monotonicity assumption:
For $\alpha, \beta \in \Lambda$ such that $\alpha \succeq \beta$,

$$R(\alpha) \subseteq R(\beta). \quad \ldots\ldots\ldots\ldots (1)$$

Upper and lower approximations:
Given a subset $Y \subseteq W$, we consider the upper approximation

$$R(\alpha)^*Y = \{w \in W : \exists y \in Y, yR(\alpha)w\} \quad \ldots (2)$$

and the lower approximation

$$R(\alpha)_*Y = \{w \in W : \forall z \in W, wR(\alpha)z \Rightarrow z \in Y\}. \quad (3)$$

where the approximations are defined for all $\alpha \in \Lambda$.

It is easily seen that

$$R(\alpha)_*Y \subseteq Y \subseteq R(\alpha)^*Y$$

and when $R(\alpha)$ is an equivalence relation, $R(\alpha)^*Y$ and $R(\alpha)_*Y$ become the standard upper and lower approximation, respectively.

We have the next proposition in view of the monotonicity.

Proposition 1. Assume that Y is an arbitrarily given subset of W. For $\alpha, \beta \in \Lambda$ such that $\alpha \succeq \beta$,

$$R(\alpha)^*Y \subseteq R(\beta)^*Y \quad \ldots\ldots\ldots\ldots (4)$$

and

$$R(\alpha)_*Y \supseteq R(\beta)_*Y. \quad \ldots\ldots\ldots\ldots (5)$$

The indexed rough approximation has a basis on a polymodal system, as we will see in the next section. Since a polymodal logic is also called a multimodal logic, and hence we call the indexed rough approximation a *multi-rough set*.

3. A Polymodal System

3.1. A Family of the Kripke Models

We use the framework of modal logic discussed in Chellas [1], except that we consider polymodal logic [8]. Namely, modal operators are denoted by

$[\alpha]$A: A is necessary with the label α,

$\langle\alpha\rangle$A: A is possible with the label α,

where $\alpha \in \Lambda$.

The above mentioned set of objects is identified with the set of possible worlds in the Kripke model. Moreover the relations $R(\alpha)$ used for the approximations are identified with the accessibility relations in the Kripke model.

Thus, the model is

$$\mathscr{M} =< W, R(\alpha), P >, \alpha \in \Lambda$$

in which the atomic sentences are $\mathbb{P}_1, \mathbb{P}_2, \ldots$ and P is the sequence $P =< P_1, P_2, \ldots >$. P_i is the subset of W in which the atomic sentence $\mathbb{P}_i$ is true.

When we consider truth or falsity of a sentence A in W, the symbol

$$\mathscr{M}, k \models \mathrm{A}$$

means that A is true at the possible world k.

The symbol $\|\mathrm{A}\|$ means the subset of possible worlds at which A is true, i.e.,

$$\|\mathrm{A}\| = \{k \in W : \mathscr{M}, k \models \mathrm{A}\}.$$

For the modal operators,

$$\mathscr{M}, k \models \langle\alpha\rangle\mathrm{A}$$

means that A is possibly true at k, which is defined by

$$\mathscr{M}, k' \models \mathrm{A}$$

for some k' such that $kR(\alpha)k'$.

$$\mathscr{M}, k \models [\alpha]\mathrm{A}$$

implies that A is necessarily true at k, which is defined by

$$\mathscr{M}, k' \models \mathrm{A}$$

for every k' such that $kR(\alpha)k'$.

The following proposition relates the indexed rough approximation and the polymodal system.

Proposition 2.

$$\mathscr{M}, k \models [\alpha]\mathrm{A} \Leftrightarrow k \in R_*(\alpha)\|\mathrm{A}\|,$$
$$\mathscr{M}, k \models \langle\alpha\rangle\mathrm{A} \Leftrightarrow k \in R^*(\alpha)\|\mathrm{A}\|.$$

3.2. An Axiomatic System and Completeness

Let us introduce an axiomatic system and consider the completeness between the above model and the axiomatic system.

We assume the following axioms:

Df[]: $[\alpha]\mathrm{A} \leftrightarrow \neg\langle\alpha\rangle\neg\mathrm{A}$

K: $[\alpha](\mathrm{A} \to \mathrm{B}) \to ([\alpha]\mathrm{A} \to [\alpha]\mathrm{B})$

T: $[\alpha]\mathrm{A} \to \mathrm{A}$

Pos: $\alpha \succeq \alpha' \Rightarrow \langle\alpha\rangle\mathrm{A} \to \langle\alpha'\rangle\mathrm{A}$,

in which the assumption **Pos** corresponds to Eq.(1). Note also that in addition to the above axioms, the standard axioms of propositional logic [1] and the next inference rules are assumed.

MP: $\dfrac{\mathrm{A}, \mathrm{A} \to \mathrm{B}}{\mathrm{B}}$,

RN: $\dfrac{\mathrm{A}}{[\alpha]\mathrm{A}}$.